Continuous-State Branching Processes: Genealogy, Duality and Interaction

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AMSS Summer School August 2022

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Introduction, references and organisation

Continuous-state branching processes (CSBPs) have been defined in the sixties (Jirina 1959, Lamperti 1967). They are the continuous-state continuous-time analogue of Galton-Watson processes and play an important role as they appear in many stochastic models, as for instance:

- Continuum random trees. Aldous (1991,...), Le Gall and Le Jan (1998), Duquesne and Le Gall (2002,...),...
- Superprocesses. Dynkin (1991), Dawson, Watanabe, Perkins, Le Gall,...
- Exchangeable coalescents. Kingman (1982), Pitman (1999), Schweinsberg (2000, 2003,...), Donnelly and Kurtz (1999), Bertoin and Le Gall (2003,...)...,...

The purpose of this short course is to shed light on two aspects of CSBPs, both relying on the concept of *stochastic duality* (which we shall specify later):

(I) The genealogy backwards in time of continuous-state branching populations. Quite active topic (Athreya, Lambert, Popovic, S.Harris, Johnston, Roberts, Abraham, Delmas, Chen, He).

Questions (Talks 1 and 2)

1. *How to follow the ancestral lineages back in time in CSBPs?*
2. *What kind of coalescent processes arises?*

(II) Imagine that besides branching, individuals are fighting by pair in the population (Competition); this is the so called Logistic CSBP (Lambert 2005).

Questions (Talk 3)

1. *Are there strong enough reproduction laws to face the competition and explosion to occur? (∞* **accessible***.)*
2. *Is it possible to start the process from ∞ ?*

Talk 1: CSBPs, Ancestral Lineages and Siegmund duality

Clément Foucart

Background on CSBPs

Definition

A CSBP is a Markov Process $(X_t, t \geq 0)$ taking values in $[0, \infty]$ satisfying the branching property, i.e. such that

$\forall x, y \geq 0, \forall t \geq 0,$

$$X_t(x+y) \stackrel{\mathscr{L}}{=} X'_t(x) + X''_t(y) \text{ (branching property)}$$

where

- $(X_t(x+y), t \geq 0)$ is the process started from $x+y$
- $(X'_t(x), t \geq 0)$ and $(X''_t(y), t \geq 0)$ are two indep. copies of the process started from x and y.

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$\Rightarrow X_t(x)$ is a positive infinitely divisible r.v.

Its semigroup takes the following form:

Theorem (Characterization: Silverstein (1968))

$$\mathbb{E}_x[e^{-qX_t}] = e^{-xu_t(q)}. \tag{1}$$

where $(u_t(q), t \geq 0)$ *is valued in* $(0, \infty)$ *and solves*

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t(q) = -\Psi(u_t(q)), \;\; u_0(q) = q \tag{2}$$

with Ψ, *called branching mechanism, of the Lévy-Khintchine form: for all* $q \geq 0$

$$\Psi(q) = -\lambda + \frac{\sigma^2}{2} q^2 + \gamma q + \int_0^{+\infty} \left(e^{-qx} - 1 + qx \mathbb{1}_{\{x \leq 1\}}\right) \pi(\mathrm{d}x) \tag{3}$$

Proposition

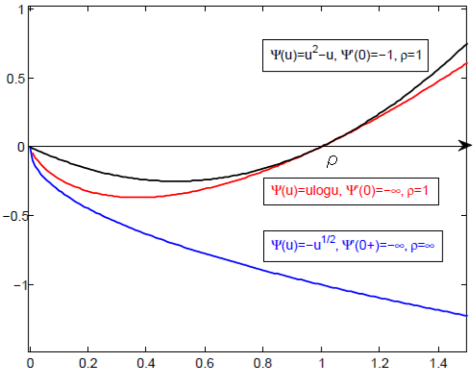
Boundaries 0 *and* ∞ *are absorbing; i.e. if one of the boundaries is reached, the process gets stuck at it.*

Classification of CSBPs

A CSBP(Ψ) is said to be

- supercritical if $\Psi'(0+) < 0$,
- critical if $\Psi'(0+) = 0$,
- subcritical if $\Psi'(0+) > 0$.

supercritical mechanisms with $\lambda = 0$



Extinction and persistence of CSBPs

Let ρ be the largest zero of Ψ, one has $u_t(q) \underset{t\to\infty}{\longrightarrow} \rho$ and

$$\text{Extinction} := \{X_t(x) \underset{t\to\infty}{\longrightarrow} 0\} \text{ has probability } e^{-x\rho}$$

- Subcritical or critical case:

  $\rho = 0$ and extinction is almost-sure.

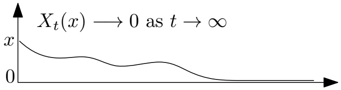
- Supercritical case:

  $\rho \in (0, \infty]$ and non-extinction has positive probability.

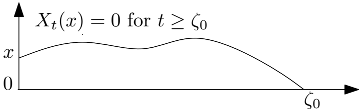
Grey's Condition:

0 is accessible (extinction in finite time) iff $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$.

Infinite decay: $\int^{\infty} \frac{du}{|\Psi(u)|} = \infty$



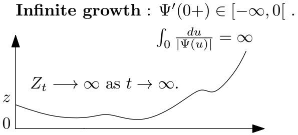
Extinction in finite time: $\int^{\infty} \frac{du}{|\Psi(u)|} < \infty$

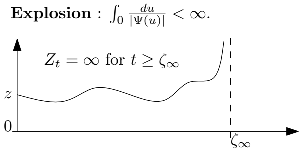


Explosion and growth of CSBP

Dynkin's condition

∞ is accessible (explosion in finite time) iff $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$.





Quick proof. Recall for all $t \geq 0, x \in [0, \infty], q > 0$,

$$\mathbb{E}_x[e^{-qX_t}] = e^{-xu_t(q)}, \quad \int_{u_t(q)}^{q} \frac{\mathrm{d}u}{\Psi(u)} = t \quad (\Leftrightarrow u_t(q) \text{ solves ode } (2)).$$

We see that

- ∞ is accessible $\Longleftrightarrow \lim_{q \to 0} u_t(q) > 0 \Longleftrightarrow \int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$.
- 0 is accessible $\Longleftrightarrow \lim_{q \to \infty} u_t(q) > 0 \Longleftrightarrow \int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$.

Quasi-stationary distribution on the non-extinction

When extinction in finite time is almost-sure, we can look for the long-term behavior of subcritical processes conditioned to stay positive

Theorem (Lambert (2006), Li (2000))

Assume $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$ *and* Ψ *subcritical. There exists a probability law* ν_∞ *on* $(0,\infty)$ *such that for all borelian* $A \subset (0,\infty)$

$$\nu_\infty(A) := \lim_{t\to\infty} \mathbb{P}(X_t(x) \in A | X_t(x) > 0).$$

The Laplace transform of ν_∞ *is*

$$\int_0^\infty e^{-uz}\nu_\infty(\mathrm{d}z) = 1 - e^{-\Psi'(0^+)\int_u^\infty \frac{\mathrm{d}x}{\Psi(x)}}.$$

CSBPs: generator

A CSBP has for generator, the operator $\mathscr{L}$ acting on any $f \in C_0^2$ as follows:

$$\begin{aligned}\mathscr{L}f(z) := & -\lambda z f(z) + \frac{\sigma^2}{2} z f''(z) - \gamma z f'(z) \\ & + z\int_0^\infty (f(z+u) - f(z) - u\mathbb{1}_{[0,1]}(u) f'(z))\pi(\mathrm{d}u) \quad (4)\end{aligned}$$

with $\lambda \geq 0, \gamma \in \mathbb{R}$, $\sigma \geq 0$, and π s.t. $\int_0^{+\infty}(1 \wedge u^2)\pi(\mathrm{d}u) < +\infty$.

- the first term $-\lambda z f(z) = \lambda z(f(\infty) - f(z))$ is seen as a jump from z to ∞ at rate λz.
- Recall

  $$\Psi(q) = -\lambda + \frac{\sigma^2}{2}q^2 + \gamma q + \int_0^{+\infty}\left(e^{-qu} - 1 + qu\mathbb{1}_{\{u\leq 1\}}\right)\pi(\mathrm{d}u)$$

  Set $e_q(z) := e^{-qz}$ for all $q \geq 0$ and $z \in [0,\infty]$. One has

  $$\mathscr{L}e_q(z) = \Psi(q) z e_q(z).$$

  Hence Ψ characterized $\mathscr{L}$.

Martingale Problem

Theorem

*Let X be a càdlàg CSBP with branching mechanism Ψ. The process X is the unique solution to the following martingale problem (**MP**) : for any function $f \in \mathcal{D}_X := \mathrm{vect}\{e_q, q \geq 0\}$*

$$t \mapsto f(X_t) - \int_0^t \mathscr{L} f(X_s)\, \mathrm{d}s \text{ is a martingale.}$$

Remark

Let $\mathcal{G}^Y$ be the generator of a spectrally positive Lévy process (spLp) Y with Laplace exponent Ψ, for any $f \in \mathcal{D}_X$,

$$\mathscr{L} f(z) = z\mathcal{G}^Y f(z).$$

Lamperti's time change

Let Y be a spLp with Laplace exponent Ψ started from x, set $\sigma_0 := \inf\{t > 0 : Y_t \leq 0\}$ and

$$X_t = \begin{cases} Y_{C_t} & 0 \leq t < \theta_\infty \\ 0 & t \geq \theta_\infty \text{ and } \sigma_0 < \infty \\ \infty & t \geq \theta_\infty \text{ and } \sigma_0 = \infty. \end{cases}$$

where $t \mapsto C_t := \inf\{u \geq 0; \theta_u > t\} \in [0, \infty]$ is the right-inverse of

$$\theta_t := \int_0^{t \wedge \sigma_0} \frac{\mathrm{d}s}{Y_s}.$$

Theorem (Volkonskii, Lamperti, Lambert et al.'s survey 2008)

$(X_t, t \geq 0)$ is a CSBP(Ψ) started from x.

Dawson-Li's Stochastic Equation & heuristics

Any CSBP(Ψ) starting from $x \in [0, \infty)$ can be seen as solution to

$$X_t = x + \sigma \int_0^t \sqrt{X_s} \mathrm{d}B_s - \gamma \int_0^t X_s \mathrm{d}s + \int_0^t \int_0^{X_{s-}} \int_{(0,1]} h \bar{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) + \int_0^t \int_0^{X_{s-}} \int_{(1,\infty]} h \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \tag{5}$$

The parameters $\gamma \in \mathbb{R}$, $\sigma \in [0, \infty)$ are the diffusive coefficients, B is a standard Brownian motion (depending on x), $\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h)$ is an independent Poisson random measure with intensity $\mathrm{d}s\mathrm{d}u\pi(\mathrm{d}h)$ (with $\pi(\infty) = \lambda$) and $\bar{\mathcal{N}}$ stands for the compensated random measure: $\bar{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) := \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) - \mathrm{d}s\mathrm{d}u\pi(\mathrm{d}h)$.

$\Rightarrow$ *Heuristically*, prior to an atom of time t of $\mathcal{N}$, an individual u is chosen uniformly in $[0, X_{t-}]$ and reproduces (leaving an amount of children h) or dies.

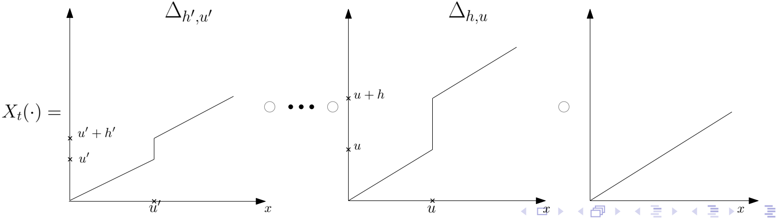
A second form for the generator of the CSBP

From the SDE (5) or directly from the first form of $\mathscr{L}$ in (4), one gets

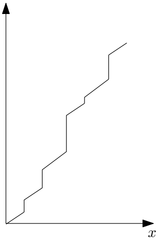
$$\mathscr{L}G(x) = \frac{\sigma^2}{2} zG''(z) - \gamma zG'(z)$$
$$+ \int_0^\infty \pi(\mathrm{d}h) \int_0^\infty \mathrm{d}u \left(G(\Delta_{h,u}(x)) - G(x) - h\mathbb{1}_{\{u\leq x\}} G'(x) \mathbb{1}_{\{h\leq 1\}} \right)$$

with $\Delta_{h,u}(x) := x + h\mathbb{1}_{\{x\geq u\}}$.

$\Rightarrow$ *heuristically* (if $\sigma = \gamma = \pi((0,1]) = 0$), then X evolves as follows



We end up *for any fixed time t* with a process $(X_t(x), x \geq 0)$ with nondecreasing sample paths

$X_t(\cdot) =$ 

whose one-dimensional laws are infinitely divisible (by the branching property):

$$\mathbb{E}[e^{-qX_t(x)}] = e^{-xu_t(q)},$$

namely $(X_t(x), x \geq 0)$ is a subordinator (i.e. a nondecreasing Lévy process) with Laplace exponent $q \mapsto u_t(q)$.

- ⚠The subordinator $X_t(\cdot)$ may be not strictly increasing and may or may not have a drift: Call ℓ_t and d_t respectively the Lévy measure and the drift of $(X_t(x), x \geq 0)$:

$$u_t(q) = d_t q + \int_{(0,\infty]} (1 - e^{-qx}) \ell_t(\mathrm{d}x).$$

  - For the sake of simplicity, we shall mainly consider the case $\Psi'(\infty) = \infty$ (infinite variation), which ensures that
$$\lim_{q\to\infty} \frac{u_t(q)}{q} = d_t = 0 \text{ for all } t.$$
  - Under Grey's condition: $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$, one has $u_t(\infty) < \infty$. Hence for all $t > 0$, $d_t = 0$ and $u_t(\infty) = \ell_t((0,\infty]) < \infty$, i.e. $X_t(\cdot)$ is a compound Poisson process.
- One interprets $(X_t(x-), X_t(x)]$ as the descendants of the initial individual x at time t. If it is empty then x has no descendant (recall $d_t = 0$).

So far, we only have given heuristics on the existence of the two-parameters process $(X_t(x), t \geq 0, x \geq 0)$. This can be made rigorously for instance

- by using Dawson-Li's SDE's framework (Dawson Li AoP 2012) with a Gaussian time-space white noise W instead of a Brownian motion so that

$$X_t = x + \sigma \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \cdots = x + \sigma \int_0^t \sqrt{X_s} \mathrm{d}B_s + \cdots$$

  for some Brownian motion B depending on x.
- or by invoking Kolmogorov's extension theorem to show the existence of a process $(X_t(x), t \geq 0, x \geq 0)$ such that $((X_t(x_1))_{t\geq 0}, (X_t(x_2) - X_t(x_1))_{t \geq 0}, \cdots)$ are indep. CSBPs(Ψ).

We now introduce a more general flow due to Bertoin and Le Gall (PTRF 2000) which keep tracks of the genealogy between individuals at different generations.

Bertoin-Le Gall's flow of subordinators

On the one hand, the Markov property of the CSBP entails that for any s and t, we have

$$u_{s+t}(\cdot) = u_s \circ u_t(\cdot)$$

On the other hand, if $X_{0,s}(\cdot)$ and $X_{s,s+t}(\cdot)$ are two independent subordinators with Laplace exponent u_s and u_t, then the process

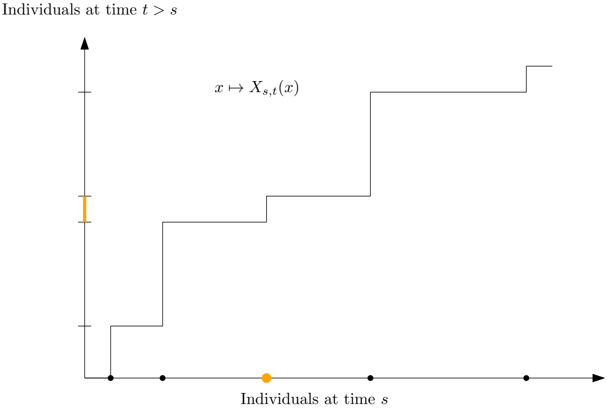
$$X_{s+t}(\cdot) := X_{s,s+t} \circ X_{0,s}(\cdot)$$

is a subordinator with Laplace exponent $u_s \circ u_t(\cdot) = u_{t+s}$. This leads Bertoin and Le Gall to show that there exists a stochastic flow $(X_{s,t}(x), s \leq t, x \geq 0)$ such that

1. $\forall s \in \mathbb{R}, x > 0$, $(X_{s,t}(x), t \geq s)$ is a CSBP(Ψ) started from x
2. $\forall s \leq t$, $x \mapsto X_{s,t}(x)$ is a subordinator with Laplace exponent $\lambda \mapsto u_{t-s}(\lambda)$
3. $\forall r \leq s \leq t$, a.s. $X_{r,t} = X_{s,t} \circ X_{r,s}$.
4. $\forall t \in \mathbb{R}$, $(X_{r,s}, r \leq s \leq t)$ and $(X_{r,s}, t \leq r \leq s)$ are indep.

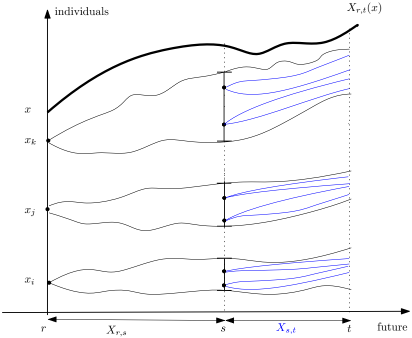
This flow allows us to consider an infinite branching population model with arbitrary old ancestors.

- The interval $[0, X_{s,t}(x)]$ gathers the descendants at time t of the population represented at time s by $[0, x]$.
- If $X_{s,t}(y-) < X_{s,t}(y)$, any individual $z \in (X_{s,t}(y-), X_{s,t}(y)]$, is a descendant of the individual y alive at time s



- The property $X_{r,t} = X_{s,t} \circ X_{r,s}$ ensures that the (forwards) genealogy is coherent in the sense that

"children of children are grandchildren":



Proof of coherence.

Let $r < s < t$. If z at time t is a descendant of y at time s, and y is a descendant of x at time r, then

$$X_{s,t}(y-) < z \leq X_{s,t}(y) \quad \text{and} \quad X_{r,s}(x-) < y \leq X_{r,s}(x).$$

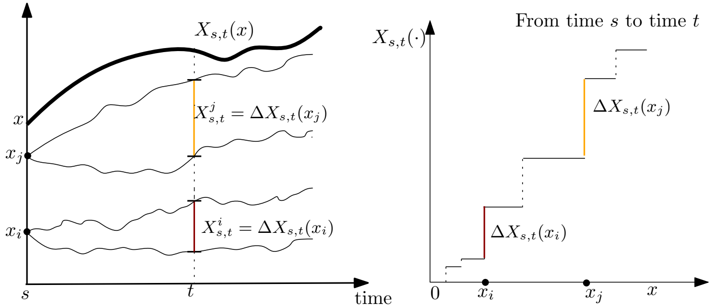
Since $X_{s,t}$ is nondecreasing and $X_{r,t} = X_{s,t} \circ X_{r,s}$:

$$X_{r,t}(x-) < z \leq X_{r,t}(x)$$

then z at time t is a descendant of x at time r.

$\square$

Schematic representation: evolution in time and families at a given time.

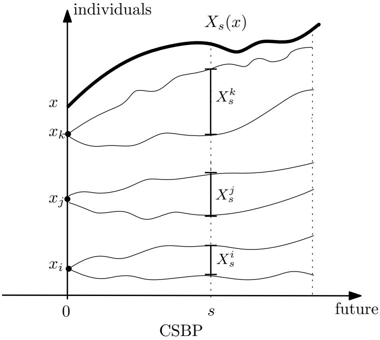


where $\Delta X_{s,t}(x) = X_{s,t}(x) - X_{s,t}(x-)$.

To sum up

- The three parameters flow $(X_{s,t}(x), -\infty < s \leq t < \infty, x \in (0,\infty))$ provides a population model with infinite size at all times and arbitrary old ancestors.
- The random partition of $(0,\infty)$ into the intervals: $\{(X_{s,t}(y-), X_{s,t}(y)], y \in J_{s,t}\}$ represents families of individuals at time t sharing ancestor at time s.
- Under Grey's condition, the subordinators $X_{s,t}$ are drift-free compound Poisson processes. In particular, the set of jump 'times' $J_{s,t}$ of $X_{s,t}$ are atoms of a Poisson process with intensity $\ell_{t-s}((0,\infty)) = u_{t-s}(\infty) < \infty$.
- Under Dynkin's condition: $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$, $u_t(0) > 0$ and the subordinators $X_{s,t}$ are thus sent to ∞ at rate $u_t(0)$. Hence there is a last interval (of infinite length) in the partition.

Question 1: How to follow the ancestral lineage of a given individual **backwards in time**?



Inverse flow : definition and first properties

We call **inverse flow** the process $(\hat{X}_{s,t}(y), s \leq t, y \geq 0)$ defined for all $s \leq t, y \geq 0$ by

$$\hat{X}_{s,t}(y) = X^{-1}_{-t,-s}(y) := \inf\{x : X_{-t,-s}(x) > y\}.$$

The individual $\hat{X}_{s,t}(y)$ is the ancestor at time $-t$ of the individual y considered at time $-s$ with $s \leq t$. We work now going backwards in time.

Lemma

1. *For all* $s \leq t$ *and* $x, y > 0$,
$$\{\hat{X}_{s,t}(y) < x\} = \{X_{-t,-s}(x) > y\} \ a.s$$
2. $\forall t \geq 0$, $(\hat{X}_{r,s}, r \leq s \leq t)$ *et* $(\hat{X}_{r,s}, t \leq r \leq s)$ *are independent.*
3. $\forall s \leq u \leq t$, *a.s.* $\hat{X}_{s,t} = \hat{X}_{u,t} \circ \hat{X}_{s,u}$

From Item 1, we get the **duality relationship** :

Lemma

$$\forall s \le t,\ \forall x, y \ge 0,\ \mathbb{P}(X_{s,t}(x) > y) = \mathbb{P}(\hat{X}_{s,t}(y) < x).$$

Definition

Let $(\hat{X}_t(x), t \ge 0) := (\hat{X}_{0,t}(x), t \ge 0)$. The process $(\hat{X}_t, t \ge 0)$ follows the ancestral lineage of individual x taken at time 0. We call it **Ancestral Lineage Process**.

It satisfies the so-called **Siegmund duality**

$$\forall t \ge 0,\ \forall x, y > 0, \mathbb{P}(\hat{X}_t(y) > x) = \mathbb{P}(X_t(x) < y). \tag{6}$$

Siegmund (AoP 1976) establishes that a one-dimensional Markov process admits a dual process satisfying (6) iff it is stochastically monotone.

Theorem

$(\hat{X}_t, t \geq 0)$ is Markov. Its semi-group $(\hat{P}_t, t \geq 0)$ satisfies for any bounded measurable f,

$$\mathbb{E}[\hat{P}_t f(\mathbb{e}_q)] = \mathbb{E}[f(\mathbb{e}_{u_t(q)})] \quad \text{for all } q > 0 \tag{7}$$

where $\mathbb{e}_q$ are $\mathbb{e}_{u_t(q)}$ exponential random variables with parameter q and $u_t(q)$.

The identity (7) caracterised $\hat{P}_t$ by the Laplace transform

$$\int_0^\infty q e^{-qy} \hat{P}_t f(y) \mathrm{d}y = \int_0^\infty u_t(q) e^{-u_t(q) y} f(y) \mathrm{d}y.$$

This is not invertible in general (but Feller and Neveu cases are...)

Proof

Proposition (Boundaries)

1. *The point* 0 *is an entrance of* $(\hat{X}_t, t \geq 0)$ *iff* $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} < \infty$ *(*0 *is an exit for* $(X_t, t \geq 0)$*).*
2. *The point* ∞ *is an entrance of* $(\hat{X}_t, t \geq 0)$ *iff* $\int_0 \frac{\mathrm{d}u}{|\Psi(u)|} < \infty$ *(*∞ *is an exit of* $(X_t, t \geq 0)$*).*
3. *The semi-group* $(\hat{P}_t, t \geq 0)$ *(extended on* $[0, \infty]$*) is Feller.*

When 0 or ∞ are entrance, the processes starting from theses points can be interpreted as follows:

1. $\hat{X}_t(0)$ is the first individual from generation t (in the past) to have descendants at time 0.
2. $\hat{X}_t(\infty)$ is the first individual from generation t (in the past) to have an infinite interval of descendants at time 0.

Long-term behaviors

Recall ρ the largest zero of Ψ, fixed point of $q \mapsto u_t(q)$

Proposition

1. *If* Ψ *is supercritical,* $(\hat{X}_t, t \geq 0)$ *is positive recurrent with stationary law* $\mathbb{e}_\rho$ $(\sim \exp(\rho))$*: for all* x*, as* $t \to \infty$
$$\hat{X}_t(x) \overset{\text{in law}}{\longrightarrow} \mathbb{e}_\rho$$
2. *If* Ψ *is sub-critical,* $\hat{X}$ *is transient (i.e. it tends to* ∞ *a.s.);*
3. *If* Ψ *is critical,* $(\hat{X}_t, t \geq 0)$ *is transient iff* $\int_0 \frac{u}{\Psi(u)}\mathrm{d}u < \infty$*, otherwise it si null recurrent.*

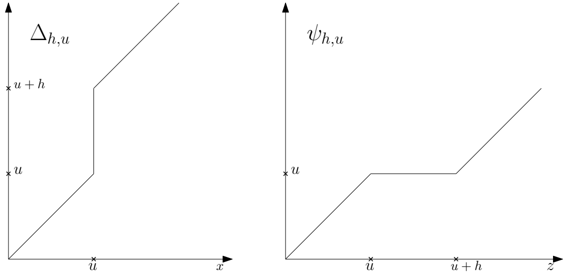
Proof of (1)

Heuristics for (2). Intuitively in the subcritical case, for any fixed level $a > 0$, all individuals below a from old generations in the past won't have descendants at time 0. Hence the ancestral lineage of x from time 0 must go above level a from a certain time.

Generator of $\hat{X}$

Let

$$\psi_{h,u}(z) := \Delta_{h,u}^{-1}(z) = z\mathbb{1}_{[0,u]}(z) + u\mathbb{1}_{[u,u+h]}(z) + (z-h)\mathbb{1}_{[u+h,\infty)}(z).$$



Set

$$\hat{\mathscr{L}}F(z) := \frac{\sigma^2}{2}zF''(z) + \left(\frac{\sigma^2}{2} + \gamma z\right)F'(z)$$
$$+ \int_0^\infty \pi(\mathrm{d}h)\int_0^\infty \mathrm{d}u\left[F(\psi_{h,u}(z)) - F(z) + h\mathbb{1}_{\{h\le 1\}}F'(z)\mathbb{1}_{\{z>u\}}\right]$$

Theorem

$\forall F \in \mathscr{D} := \left\{F \in C_0^2; F' \in L^1 \text{ and } \beta xF'(x), \frac{\sigma^2}{2}xF''(x) \underset{x\to\infty}{\longrightarrow} 0\right\}.$

$$(\mathrm{MP}) \quad \left(F(\hat{X}_t(y)) - \int_0^t \hat{\mathscr{L}}F(\hat{X}_s(y))\mathrm{d}s, t \ge 0\right) \text{ is a martingale}$$

and the martingale problem is well posed.

Generator in Courrège form

Replacing $\psi_{h,u}$ by its expression and integrating yields the following more explicit or usual form

Lemma

For all F,

$$\hat{\mathscr{L}} F(z) = \frac{\sigma^2}{2} z F''(z) + b(z) F'(z) + \int_0^z \left[F(z-h) - F(z) + h F'(z) \right] \nu(z, \mathrm{d}h)$$

with

$$\nu(z, \mathrm{d}h) := \mathbb{1}_{\{h \le z\}} \left((z-h)\pi(\mathrm{d}h) + \bar{\pi}(h)\mathrm{d}h \right)$$

and

$$b(z) := \int_0^\infty h \left(z \mathbb{1}_{\{h \le 1\}} \pi(\mathrm{d}h) - \nu(z, \mathrm{d}h) \right) + \gamma z + \frac{\sigma^2}{2}.$$

The proof of the theorem being a bit lengthy and technical, let's try to explain it directly from the article. This will end Talk 1.

The proof of the theorem being a bit lengthy and technical, let's try to explain it directly from the article. This will end Talk 1.

Thanks!

Talk 2: Genealogy and Coalescent in CSBPs

Clément Foucart

We have seen previously that ancestral lineages can be tracked back via a Feller process with negative jumps $(\hat{X}_t, t \geq 0)$. By definition, for any couple of individuals $x \neq y$, if there is a time T such that $\hat{X}_T(x) = \hat{X}_T(y)$, then $\hat{X}_t(x) = \hat{X}_t(y)$ for all $t \geq T$ a.s. Hence $\hat{X}$ is a flow of coalescing Markov processes with negative jumps.

Question 2: How mergings occur ?

Main object of study

Let $(J_i^\lambda, i \geq 1)$ be the sequence of arrival times in a Poisson process, independent from $\hat{X}$, with intensity λ fixed. We define the partition of $\mathbb{N}$ $C^\lambda(t)$ by

$$i \overset{C^\lambda(t)}{\sim} j \text{ iff } \hat{X}_t(J_i^\lambda) = \hat{X}_t(J_j^\lambda).$$

In other words, two integers i and j are in the same block of $C^\lambda(t)$ if the individuals J_i^λ and J_j^λ have a common ancestor at time t. The parameter λ can be seen as controlling the "mesh parameter" of the discretization.
$\Rightarrow$ $(C^\lambda(t), t \geq 0)$ is a certain *non-exchangeable coalescent* process. Namely for any $s > 0$, $C^\lambda(t+s)$ is obtained by merging blocks of $C^\lambda(t)$ in a certain way. We study now this process.

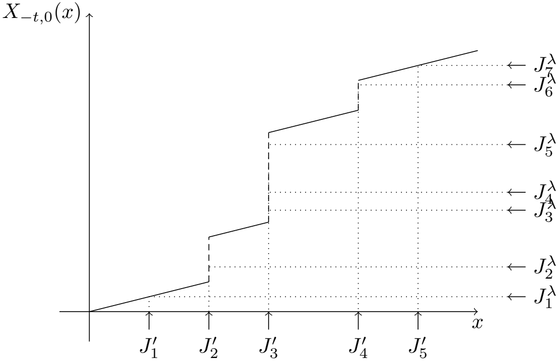


Figure: $C^\lambda_{|[7]}(t) = \{\{1\},\{2\},\{3,4,5\},\{6\},\{7\}\}$

Consecutive Partition and Coagulation Operator

A consecutive partition of $[n] := \{1, 2, \cdots, n\}$, $C = (C_1, C_2, \cdots)$ is a partition whose blocks $(C_i, i \geq 1)$ are made of *consecutive* integers. We denote by $\mathcal{C}_n$ the space of consecutive partitions of $[n]$ with $n \in \mathbb{N} \cup \{\infty\}$.

- Blocks of C are ordered by their least elements:
$$\min C_1 \leq \min C_2 \leq \cdots$$
- C is characterized by the sequence of blocks sizes
$$(\# C_1, \# C_2, \ldots).$$
- Let $C_{|[k]}$ be the restriction of C to $[k]$; for all $m \leq k$,
$$(C_{|[k]})_{|[m]} = C_{|[m]} \in \mathcal{C}_m$$
- **Coagulation operator**: Let $C \in \mathcal{C}_n$ and $C' \in \mathcal{C}_{n'}$ such that $\# C \leq n'$, define $\mathrm{Coag}(C, C')$ by
$$\mathrm{Coag}(C, C')_j := \bigcup_{i \in C'_j} C_i \quad \text{for all } j \in \mathbb{N}.$$
Note that $\mathrm{Coag}(C, C') \in \mathcal{C}_n$.

Consecutive Coalescent : definition

Example: $C = \{\{1\},\{2,3,4\},\{5,6\}\}$, $C' = \{\{1\},\{2,3\}\}$, then

$$\mathrm{Coag}(C, C') = \{\{1\},\{2,3,4,5,6\}\}.$$

Definition

A Markov process $(C(t), t \geq 0)$ valued in $\mathcal{C}_\infty$ is a consecutive coalescent if its semigroup verifies: $\forall t, s \geq 0$

$$C(t+s) = \mathrm{Coag}(C(t), C')$$

where C' is a consecutive partition, independent from $C(t)$, whose blocks sizes are i.i.d. and whose law may depend on s and t.

- The coalescent is homogeneous if the law of C' depends only on s.

Consecutive Coalescent: construction

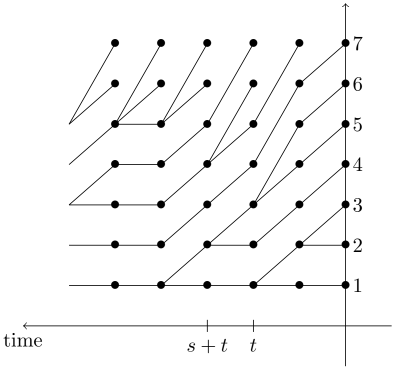
Lemma (Coagulation rate)

The coalescences in an homogeneous coalescent can be described as follows: let μ be a finite measure on $\mathbb{N}$, with $\mu(\{1\}) = 0$:

- *To each block j of C, we associate a family $(\mathbb{e}_{j,k}, k \geq 2)$ of exponential clocks with parameters $(\mu(k), k \geq 2)$.*
- *When the clock $\mathbb{e}_{j,k}$ rings, the consecutive blocks j, $j+1$, … , $j+k-1$ of C merge.*

We call μ the coagulation rate measure.

Consecutive coalescents appear directly when we reverse time in an infinite forest of immortal Galton-Watson processes in continuous time...



$$C_{|[7]}(t) = \{\{1,2,3\},\{4\},\{5,6\},\{7\}\}$$

$$C'_{|[7]} = \{\{1\},\{2,3\},\{4\},\{5,6\},\{7\}\}$$

$$\begin{aligned} C_{|[7]}(s+t) &= \mathrm{Coag}(C_{|[7]}(s), C'_{|[7]}) \\ &= \{\{1,2,3\},\{4,5,6\},\{7\}\} \end{aligned}$$

Figure: Monotone labelling of a forest of immortal Galton-Watson processes started from $-\infty$ and its coalescent in reversed time

Consecutive Coalescents in the inverse flow

Recall $(J_i^\lambda, i \geq 1)$ and $C^\lambda(t)$:

$$i \overset{C^\lambda(t)}{\sim} j \text{ iff } \hat{X}_t(J_i^\lambda) = \hat{X}_t(J_j^\lambda).$$

Theorem

The process $(C^\lambda(t), t \geq 0)$ is a consecutive coalescent (possibly inhomogeneous). Its semi-group is Feller and its one-dimensional laws are caracterised by

$$\mathbb{E}[z^{\# C_1^\lambda(t)}] = 1 - \frac{u_t(\lambda(1-z))}{u_t(\lambda)} \text{ pour tout } z \in [0,1].$$

The coagulation rate measure at time t is, for all $k \geq 2$

$$\mu_t^\lambda(k) := \frac{\sigma^2}{2} u_t(\lambda) 1_{\{k=2\}} + u_t(\lambda)^{k-1} \int_{(0,\infty)} \frac{x^k}{k!} e^{-u_t(\lambda)x} \pi(\mathrm{d}x).$$

Road map of the proof

1. Lemma 1 describes $C^\lambda(t)$ completely and states a crucial independence property
2. Lemma 2 shows that $(C^\lambda(t), t \geq 0)$ is a consecutive coalescent
3. Lemma 3 provides the coagulation rate
4. Lemma 4 explains how coalescences occur.

Poisson-box

We use a specific class of consecutive partitions, that we call Poisson boxes; they play a similar role as Kingman's paintboxes in exchangeable coalescents. Let ϕ be the Laplace exponent of a subordinator

$$\phi : \mu \mapsto d\mu + \int_{(0,\infty]} \left(1 - e^{-\mu x}\right) \ell(\mathrm{d}x).$$

Definition

A (λ, ϕ)-Poisson-box is a random consecutive partition C obtained by

$$i \overset{C}{\sim} j \iff X^{-1}(J_i^\lambda) = X^{-1}(J_j^\lambda),$$

where X is a subordinator with Laplace exponent ϕ and $(J_j^\lambda, j \geq 1)$ are atoms (i.e. arrival times) of a Poisson process with intensity λ, indep. from X.

Proposition (Key proposition)

Let C be a (λ, ϕ)-Poisson-box built from X and $(J_k^\lambda, k \geq 1)$.

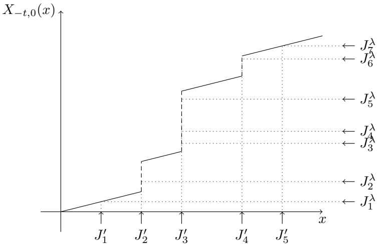
1. *C is a consecutive partition with i.i.d blocks sizes such that*

$$\mathbb{P}(\#C_1 = k) := \frac{1}{\phi(\lambda)} \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \ell(\mathrm{d}x) + d\mathbb{1}_{k=1}$$
$$= (-1)^{k-1} \frac{\lambda^k}{k!} \frac{\phi^{(k)}(\lambda)}{\phi(\lambda)},$$

   For any $s \in [0,1]$, $\mathbb{E}(s^{\#C_1}) = 1 - \frac{\phi(\lambda(1-s))}{\phi(\lambda)}$.

2. *Set for all $i \geq 1$, $J'_i := X^{-1}(J_k)$ for $k \in C_i$. The rv's $(J'_i, i \geq 1)$ are atoms of a Poisson process with intensity $\phi(\lambda)$.*
3. *$(J'_i, i \geq 1)$ and C are independent.*

Lemma 1: how $C^\lambda(t)$ looks like?



Lemma (Key Lemma=application of key prop.)

1. *$C^\lambda(t)$ is a (λ, u_t)-Poisson box, in particular* $\mathbb{E}(z^{\#C_1^\lambda(t)}) = 1 - \frac{u_t(\lambda(1-z))}{u_t(\lambda)}$
2. *$\forall i \geq 1$, set $J'_i := X^{-1}_{-t,0}(J^\lambda_k) = \hat{X}_t(J^\lambda_k)$ for $k \in C_i$. The $(J'_i, i \geq 1)$ are atoms (i.e. arrival times) of a Poisson process of intensity $u_t(\lambda)$.*
3. *$(J'_i, i \geq 1)$ and C^λ are independent.*

Lemma 2: coalescent structure

Recall the properties for all $s, t \geq 0$ and $\lambda \in (0, \infty)$:

$$u_{t+s}(\lambda) = u_t \circ u_s(\lambda),\ \hat{X}_{t+s} = \hat{X}_{t,t+s} \circ \hat{X}_t.$$

Lemma

For any $s, t \geq 0$

$$C^\lambda(t+s) = \mathrm{Coag}(C^\lambda(t), C^\lambda(t, t+s)) \tag{8}$$

where $C^\lambda(t, t+s)$ is a certain $(u_t(\lambda), u_s)$-Poisson box which is independent of $C^\lambda(t)$.

Proof

For any $s, t \geq 0$ and all $l \geq 1$, set $J_l^\lambda(t) := \hat{X}_t(J_i^\lambda)$ for all $i \in C_l^\lambda(t)$. Let $C^\lambda(t, t+s)$ be the random consecutive partition defined by

$$l \overset{C^\lambda(t,t+s)}{\sim} k \text{ if and only if } \hat{X}_{t,t+s}(J_l^\lambda(t)) = \hat{X}_{t,t+s}(J_k^\lambda(t)).$$

Let $i, j \in \mathbb{N}$. Set k and l such that $i \in C_k^\lambda(t)$ and $j \in C_l^\lambda(t)$. Recall that $\hat{X}_{t+s} = \hat{X}_{t,t+s} \circ \hat{X}_t$. This ensures that

- $\hat{X}_{t,t+s}(J_k^\lambda(t)) = \hat{X}_{t,t+s}(J_l^\lambda(t))$ holds if and only if $i \overset{C^\lambda(t+s)}{\sim} j$
- By construction, this is nothing else than

$$i \overset{\mathrm{Coag}(C^\lambda(t), C^\lambda(t,t+s))}{\sim} j$$

Lemma 3: Coagulation rate's computation

Lemma

For any $z \in (0,1)$,

$$\frac{1}{s}\left(\mathbb{E}[z^{\# C_1^\lambda(t,t+s)}] - z\right) \underset{s\to 0}{\longrightarrow} \frac{\Psi(u_t(\lambda)(1-z)) - (1-z)\Psi(u_t(\lambda))}{u_t(\lambda)}$$

$$= \sum_{k\geq 2} z^k \mu_t^\lambda(k),$$

with

$$\mu_t^\lambda(k) := \frac{\sigma^2}{2} u_t(\lambda) 1_{\{k=2\}} + u_t(\lambda)^{k-1} \int_{(0,\infty)} \frac{x^k}{k!} e^{-u_t(\lambda)x} \pi(\mathrm{d}x).$$

Lemma 4: Coalescent evolution

Let $n \geq 1$. Conditionally given $\# C^{\lambda}_{|[n]}(t-) = m$, for any $j \leq m-1$, consider the consecutive partitions of $[m]$

- $C^{j,k}_{\mathrm{in}} := (\{1\}, \ldots, \{j, \ldots, j+k-1\}, \ldots, \{m\})$ for any $2 \leq k \leq m-j$ and attach to each C^{j}_{in} a random clock $\zeta^{j,k}_{\mathrm{in}}$ with law

$$\mathbb{P}(\zeta^{j,k}_{\mathrm{in}} > s) = \exp\left(- \int_0^s \mu^{\lambda}_r(k)\mathrm{d}r \right).$$

- $C^{j}_{\mathrm{out}} := (\{1\}, \ldots, \{j, \ldots, m\})$ and attach to each C^{j}_{out} a random clock $\zeta^{j,k}_{\mathrm{out}}$ with law

$$\mathbb{P}(\zeta^{j,k}_{\mathrm{out}} > s) = \exp\left(- \int_0^s \bar{\mu}^{\lambda}_r(m-j+1)\mathrm{d}r \right).$$

Then the process jumps from $C_{|[n]}(t-)$ to $\mathrm{Coag}(C_{|[n]}(t-), D)$ with D the partition in $\{C^{j,k}_{\mathrm{in}}, C^{j}_{\mathrm{out}}\}$ associated with the first random clock that rings.

We now give some further properties of $(C^\lambda(t), t \geq 0)$ and $(\hat{X}_t, t \geq 0)$:

1. Long term behaviors?
2. "$\lambda \to \infty$"?

Corollary ("Sampling the immortals")

If Ψ is supercritical and $\lambda = \rho$. The process $(C^{\rho}(t), t \geq 0)$ is an **homogeneous** *consecutive coalescent and its coagulation rate is*

$$\mu^{\rho}(k) := \frac{\sigma^2}{2}\rho 1_{\{k=2\}} + \rho^{k-1}\int_{(0,\infty)} \frac{x^k}{k!}e^{-\rho x}\pi(\mathrm{d}x).$$

Proposition (Long term behaviors)

- *If Ψ is critical or supercritical then a.s.*

$$C^{\lambda}(t) \underset{t\to\infty}{\longrightarrow} 1_{\mathbb{N}} := (\mathbb{N}, \emptyset, \cdots) \text{ a.s.}$$

- *If Ψ is subcritical, then a.s*

$$C^{\lambda}(t) \underset{t\to\infty}{\longrightarrow} C^{\lambda}(\infty) \text{ a.s.}$$

where $\mathbb{E}[z^{\# C_1^{\lambda}(\infty)}] = 1 - e^{-\Psi'(0^+)\int_{\lambda(1-z)}^{\lambda}\frac{\mathrm{d}u}{\Psi(u)}}$. The sampled individuals $(J_1^{\lambda}, J_2^{\lambda}, \ldots)$ are thus partitioned into different ancestral families.

Complete genealogy of the population under Grey's condition

The coalescent $(C^{\lambda}(t), t \geq 0)$ only represents the genealogy of a subpopulation. *Can we describe the complete genealogy?* Assume

$$\int^{\infty} \frac{\mathrm{d}x}{\Psi(x)} < \infty.$$

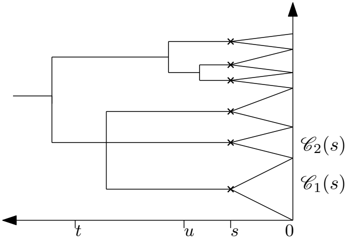
- $(X_{-s,0}(x), x \geq 0)$ is a compound Poisson process with Lévy measure $\ell_s(\mathrm{d}x)$. Its jump times, $(J_i^{u_s(\infty)})_{i\geq 1}$ are atoms of a Poisson process with intensity
$$u_s(\infty) = \ell_s((0,\infty]) < \infty.$$
- Let $(C(s,t), t \geq s > 0)$ be defined by
$$i \overset{C(s,t)}{\sim} j \text{ iff } \hat{X}_{s,t}(J_i^{u_s(\infty)}) = \hat{X}_{s,t}(J_j^{u_s(\infty)}).$$

$(C(s,t), t \geq s)$ describes the genealogy of initial individuals whose most recent common ancestors are found at time s.

Theorem

For all $s > 0$, $(C(s,t), t > s)$ is a consecutive coalescent with coagulation rate $(\mu_t^\infty, t > s)$: for all $k \geq 2$

$$\mu_t^\infty(k) := \frac{\sigma^2}{2} u_t(\infty) 1_{\{k=2\}} + u_t(\infty)^{k-1} \int_{(0,\infty)} \frac{x^k}{k!} e^{-u_t(\infty)x} \pi(\mathrm{d}x).$$



Intervals at time s are given by $(\mathscr{C}_i(s), i \in [6])$

$$C(s,u)_{|[6]} = (\{1\},\{2\},\{3\},\{4,5\},\{6\})$$
$$C(u,t)_{|[5]} = (\{1,2,3\},\{4,5\})$$
$$C(s,t)_{|[6]} = \mathrm{Coag}(C(s,u),C(u,t))_{|[6]}$$
$$= (\{1,2,3\},\{4,5,6\})$$

Figure: Schematic representation of the genealogy

Theorem

Assume Grey's condition. Let $(\mathscr{C}(t), t > 0)$ *be the process valued in* $\mathcal{C}_{\mathbb{R}^+}$ *defined by:*

$$\mathscr{C}(t) = \{(X_{-t,0}(x-), X_{-t,0}(x)], x \in J_{-t}\}$$

where J_{-t} *is the set of jump "times" of* $(X_{-t,0}(x), x \geq 0)$.

- *The process* $(\mathscr{C}(t), t > 0)$ *is an inhomogeneous Markov process such that for all* $t \geq s > 0$,
$$\mathscr{C}(t) = \mathrm{Coag}(\mathscr{C}(s), C(s,t)) \ \textit{a.s.}$$
- *In the critical or supercritical case,* $\mathscr{C}(t) \underset{t\to\infty}{\longrightarrow} \mathbb{1}_{(0,\infty)} := \{(0,\infty), \emptyset, \cdots\}$ *a.s.*
- *In the subcritical case,* $\mathscr{C}(t) \underset{t\to\infty}{\longrightarrow} \mathscr{C}(\infty)$ *a.s. and* $|\mathscr{C}_1(\infty)|$ *has for law* ν_∞*, the quasi-stationary distribution of the CSBP* **conditioned on the non-extinction**.

- The process $(C(s,t), t > s)$ describes the semigroup of the coalescent process $(\mathscr{C}(t), t > 0)$.
- Grey's condition ensures somehow a coming down from a continuous world $\mathscr{C}$ to a discrete one $\mathcal{C}$. When Grey's condition does not hold, the situation is more involved, as all individuals always have descendants.

We now explain some results in the subcritical case. Recall that in this case, $\hat{X}$ is transient, i.e. goes to ∞ a.s.. We find an almost-sure renormalisation of the inverse flow and interpret the limiting process.

Theorem (Almost sure renormalisation)

Assume $\Psi'(0+) > 0$. *Fix* $q \in (0, \infty)$. *Then,* **almost surely**

$$u_t(q)\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \hat{W}^q(x), \forall x \notin J^\lambda := \{x > 0 : \hat{W}^q(x) > \hat{W}^q(x-)\},$$

where $(\hat{W}^q(x), x \geq 0)$ *is the inverse of a subordinator, with no drift and whose Laplace exponent is* $\kappa_q : \theta \mapsto e^{-\Psi'(0+)\int_\theta^q \frac{\mathrm{d}u}{\Psi(u)}}$.

Proposition ($L \log L$ condition and exponential escape)

$$u_t(q) \underset{t\to\infty}{\sim} c_q e^{-\Psi'(0+)t} \text{ iff } \int^\infty u \log u \, \pi(\mathrm{d}u) < \infty.$$

Moreover one always has $u_t(q)/u_t(q') \underset{t\to\infty}{\longrightarrow} c_{q,q'} \in (0, \infty)$

The theorem covers the two cases with or without Grey's condition:

i) **Under Grey's condition**, one can take $q = \infty$, $u_t(\infty)\hat{X}_t \underset{t\to\infty}{\longrightarrow} \hat{W}^\infty$ a.s, and $\hat{W}^\infty$ is an inverse compound Poisson process with jump-law ν_∞, the quasi-stationary distribution of the CSBP conditioned on non-absorption.

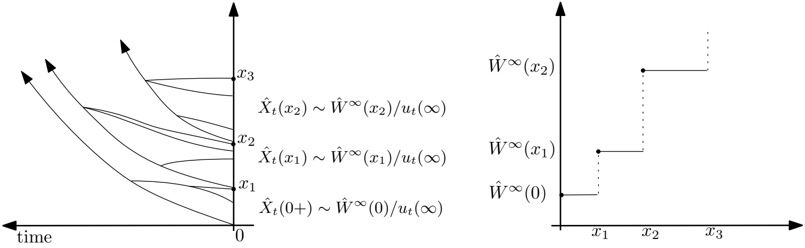


Figure: Schematic representation under Grey's condition

ii) **Without Grey's condition**: $\forall q \in (0, \infty)$, $\hat{W}^q$ has singular continuous paths (the Lévy measure is **infinite**).

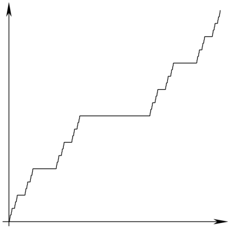


Figure: Cantor function. Source: Wikipedia

Example

Let $\gamma > 0$ and $\Psi(u) := \gamma(u+1)\log(u+1)$. $\Psi'(0+) = \gamma > 0$ and $\int^{\infty} \frac{\mathrm{d}u}{\Psi(u)} = \infty$, $u_t(q) \underset{t\to\infty}{\sim} c_q e^{-\Psi'(0+)t}$.

$$e^{-\Psi'(0+)t}\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \hat{W}(x) \text{ for all } x \geq 0 \text{ almost surely}$$

where $\hat{W}$ is an inverse Gamma subordinator.

Interpretation of $\hat{W}^q$:

define a random equivalence relation $\mathscr{A}$ on $(0,\infty)$ via

$$x \overset{\mathscr{A}}{\sim} y \text{ if and only if } \hat{W}^q(x) = \hat{W}^q(y).$$

Actually, $\mathscr{A}$ does not depend on q and corresponds to families at generation 0 having a common ancestor:

Theorem ($\mathscr{A}$ = ancestral partition)

For any $x, y \in (0,\infty)$,

$$x \overset{\mathscr{A}}{\sim} y \textit{ if and only if } \hat{X}_t(x) = \hat{X}_t(y) \textit{ for some } t \geq 0.$$

i) **Under Grey's condition**, the partition $\mathscr{A}$ is constituted of i.i.d. families with lengths of law ν_∞,

$$\mathscr{A} = \big((0, x_1], (x_1, x_2], \dots\big) \text{ a.s.},$$

where $(x_i, i \geq 1)$ is a renewal process with jump law ν_∞ (the qsd).

ii) **Without Grey's condition**, the ancestral families in $\mathscr{A}$ are separated by points $x_i, i \in I$, in the support $\mathscr{S}$ of the singular random measure $\mathrm{d}\hat{W}^\lambda$.

Proposition

Set $\Psi'(\infty) := \lim_{u\to\infty} \frac{\Psi(u)}{u} \in (0,\infty]$.

$$\dim_H(\mathscr{S} \cap [0,x]) = \frac{\Psi'(0+)}{\Psi'(\infty)} \in [0,1) \ a.s. \tag{9}$$

Example

Let Ψ be the branching mechanism with drift $\gamma = 1$ and Lévy measure $\pi(\mathrm{d}x) = x^{-\alpha-1}e^{-x}\mathrm{d}x$ with $\alpha \in (0,2)$.

i) If $\alpha \in (1,2)$, then $\Psi'(\infty) = \infty$, and $\dim_H(\mathscr{S} \cap [0,x]) = 0$ a.s.

ii) If $\alpha \leq 1$, then $\Psi'(\infty) \leq \infty$, $\dim_H(\mathscr{S} \cap [0,x]) = \frac{1}{1+\Gamma(1-\alpha)}$ a.s.

Study of the almost-sure convergence of $u_t(q)\hat{X}_t(x)$ as $t \to \infty$.

How to find martingales? Neither the semigroup nor the generator of $\hat{X}$ appear easy to handle ...

First, a general result

Let X be a stochastically monotone Markov process:

$$y \geq x \Longrightarrow \mathbb{P}(X_t(y) > z) \geq \mathbb{P}(X_t(x) > z)$$

Set P_t its semigroup and $\hat{P}_t$ that of its Siegmund dual $\hat{X}$:

$$\mathbb{P}(\hat{X}_t(x) > y) = \mathbb{P}(x > X_t(y)).$$

Theorem (Invariant functions of $\hat{X}$)

If μ_θ is a measure on $[0,\infty)$ s.t. $\mu_\theta P_t = e^{\theta t}\mu_\theta$, then

$$f_\theta : x \mapsto \mu_\theta([0,x)) \text{ is } \theta\text{-invariant for } \hat{X}, \text{ i.e. } \hat{P}_t f_\theta = e^{\theta t} f_\theta$$

so that $\left(e^{-\theta t} f_\theta(\hat{X}_t(x)), t \geq 0\right)$ is a martingale.
If $\hat{X}$ has no positive jumps and μ_θ is finite on $[0,x)$ for all $x > 0$, then

$$\mathbb{E}_x[e^{-\theta \hat{T}_y}] = \frac{\mu_\theta([0,x))}{\mu_\theta([0,y))} \text{ where } \hat{T}_y := \inf\{t > 0 : \hat{X}_t > y\} \qquad (10)$$

Proof of the general result

Set $f_\theta(x) = \mu_\theta([0,x))$ for all $x > 0$. For any $x \in (0,\infty)$ and any $t \geq 0$,

$$\begin{aligned}\hat{P}_t f_\theta(x) = \mathbb{E}\left[f_\theta\left(\hat{X}_t(x)\right)\right] &= \mathbb{E}\left[\int \mathbb{1}_{\{\hat{X}_t(x)>y\}}\mu_\theta(\mathrm{d}y)\right] \\ &= \mathbb{E}\left[\int \mathbb{1}_{\{x>X_t(y)\}}\mu_\theta(\mathrm{d}y)\right] \text{ by the duality} \\ &= \mu_\theta P_t\big([0,x)\big) = e^{\theta t}\mu_\theta\big([0,x)\big) = e^{\theta t} f_\theta(x).\end{aligned}$$

Hence by the Markov property, $(e^{-\theta t} f_\theta(\hat{X}_t), t \geq 0)$ is a martingale.

The map $f_\theta : x \mapsto \mu_\theta([0,x))$ is left-continuous and by the bounded optional stopping time theorem at time $t \wedge \hat{T}_y$:

$$\mathbb{E}\left[e^{-\theta t\wedge \hat{T}_y} f_\theta\left(\hat{X}_{t\wedge \hat{T}_y}(x)\right)\right] = f_\theta(x).$$

Since f_θ is non-decreasing and $\hat{X}_{t\wedge \hat{T}_y}(x) \leq y$ a.s, one has for any $t \geq 0$, $f_\theta\left(\hat{X}_{t\wedge \hat{T}_y}(x)\right) \leq f_\theta(y)$. On the event $\{\hat{T}_y < \infty\}$, the left-continuity of f_θ and the absence of negative jumps in the process $(\hat{X}_t, t \geq 0)$ ensure that $f_\theta\left(\hat{X}_{t\wedge \hat{T}_y}(x)\right) \underset{t\to\infty}{\longrightarrow} f_\theta(y)$. This yields

$$f_\theta(x) = \lim_{t\to\infty} \mathbb{E}\left[e^{-\theta t\wedge \hat{T}_y} f_\theta\left(\hat{X}_{t\wedge \hat{T}_y}(x)\right)\right] = \mathbb{E}\left[e^{-\theta \hat{T}_y} f_\theta(y) \mathbb{1}_{\{\hat{T}_y<\infty\}}\right],$$

which provides the identity :

$$\mathbb{E}_x[e^{-\theta \hat{T}_y}] = \frac{\mu_\theta([0,x))}{\mu_\theta([0,y))} \text{ where } \hat{T}_y := \inf\{t > 0 : \hat{X}_t > y\} \quad (11)$$

Sketch of proof

1. Study of a θ-invariant measure μ_θ for the CSBP:
   - Let $\mathscr{L}$ be the generator of X:

$$\mu_\theta \text{ is } \theta\text{-invariant} \iff \mu_\theta \mathscr{L} = \theta \mu_\theta$$
$$\iff c_\theta(q) := \langle \mu_\theta, e^{-q\cdot} \rangle \text{ solves } -\Psi(q) c_\theta'(q) = \theta c_\theta(q).$$

   - The solution of the ode is $c_\theta(q) = \exp\left(-\theta \int_1^q \frac{\mathrm{d}u}{\Psi(u)}\right) \leftarrow$ this is completely monotone on $(0, \infty)$ and by Bernstein-Widder's theorem :

$$\exists \text{ a Borel measure } \mu_\theta \text{ s.t. } c_\theta(q) = \int_{[0,\infty)} e^{-qx} \mu_\theta(\mathrm{d}x).$$

2. By our previous theorem: $f_\theta(x) := \mu_\theta([0, x))$ is θ-invariant for $\hat{X}$ and $\lim_{t\to\infty} e^{-\theta t} f_\theta(\hat{X}_t(x))$ exists a.s. (martingale convergence theorem).
   - c_θ is regularly varying at 0, by a Tauberian theorem

$$f_\theta(x) = \mu_\theta([0, x)) \underset{\infty}{\sim} \Gamma\left(1 + \theta/\Psi'(0+)\right)^{-1} c_\theta(1/x) \quad (\star).$$

   - An analysis of the r.h.s in $(\star)$ and the almost-sure convergence above provide $u_t(q)\hat{X}_t(x) \underset{t\to\infty}{\longrightarrow} \hat{W}^\lambda(x)$ a.s.

$$\text{Study of } \mathscr{A} : x \sim y \iff \hat{W}^q(x) = \hat{W}^q(y).$$

Recall $\hat{W}^q(x) = \lim_{t\to\infty} u_t(q)\hat{X}_t(x)$. We use our previous result with the discretized population $(J_i^\lambda, i \geq 0)$.

1. Clearly if x and y have a common ancestor then $\hat{X}_t(x) = \hat{X}^t(y)$ from some t, hence $\hat{W}^q(x) = \hat{W}^q(y)$ and $x \overset{\mathscr{A}}{\sim} y$.
2. Define now $i \overset{\mathscr{A}^\lambda}{\sim} j$ iff $\hat{W}^q(J_i^\lambda) = \hat{W}^q(J_j^\lambda)$, then by applying the key proposition, we get the law of block sizes and it happens that it coincides with that of $C^\lambda(\infty)$. The latter corresponds to the ancestral partition for the sample $(J_i^\lambda, i \geq 1)$.
3. Let $x \overset{\mathscr{A}}{\sim} y$, s.t. $x \neq y$. Thus $\exists u > 0; x, y \in (W^q(u-), W^q(u)]$ with $W^q := (\hat{W}^q)^{-1}$. One achieves the proof by a coupling argument, with λ large enough so that

$$W^q(u-) < J_i^\lambda < x < y \leq J_j^\lambda < W^q(u)$$

and $\hat{W}^q(J_i^\lambda) = \hat{W}^q(J_j^\lambda) = u$, hence all individuals in $(J_i^\lambda, J_j^\lambda]$, in particular x and y, share a common ancestor, and finally $\mathscr{A}$ is the ancestral partition.

Thanks!

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