Galton-Watson trees and Lévy trees.

I) Scaling limits and charaterizations.

II) cut tree.

Hui He (BNU).

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hehui@bnu.edu.cn

I) Scaling limits

① Galton - Watson process (Galton - Watson (1871).)

$Z_0 = 1.$ $\quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{i,n}$ $\quad n = 0, 1, 2, \cdots$

$\{\xi_{i,n},\ i \geq 1,\ n \geq 1\}$ i.i.d. $\overset{(d)}{\sim} \underbrace{\{p_k : k = 0, 1, 2, \cdots\}}_{\text{offspring law.}}$

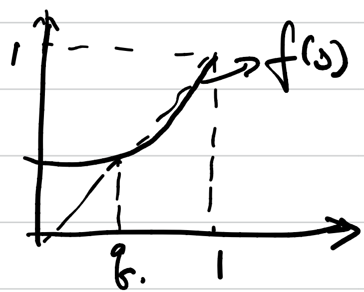
Reference book: Athreya & Ney (1971)

③ Extinction probability.

- If $m := \sum_{k=1}^{+\infty} k P_k \le 1$, then $P(\lim_{n\to\infty} Z_n = 0) = 1$.
- If $m > 1$, then $\underbrace{P(\lim_{n\to\infty} Z_n = \infty)}_{\vdots} = 1 - \underbrace{P(\lim_{n\to\infty} Z_n = 0)}_{=q} > 0$.

q is the minimum solution

of $s = f(s) = \sum_{k=1}^{+\infty} s^k \cdot P_k$, $0 \le s \le 1$.



③ Galton-Watson tree

Neveu (1986): $\mathcal{W} = \bigcup_{n=0}^{+\infty} (\mathbb{N}^*)^{\otimes n}$ $\quad \mathbb{N}^* = \{1, 2, \dots\}$

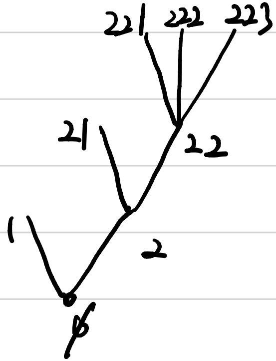
$(\mathbb{N}^*)^{\otimes 0} = \phi.$

Def: A rooted ordered tree t is a subset of $\mathcal{W}$ such that.

i). $\phi \in t$

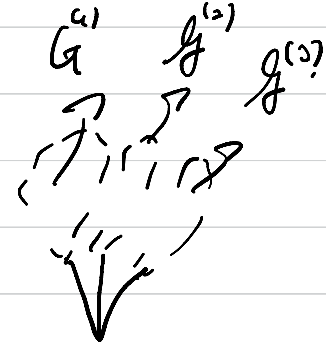
ii) $w \in t \setminus \{\phi\} \Rightarrow \pi(w) \in t.$ $[\, w = (w_1, \dots, w_{n-1}, w_n),\ \underbrace{\pi(w) = (w_1, \dots, w_{n-1})}_{\text{father of } w.}\,]$

iii) $\forall\, w \in t$, $\exists$ a finite integer $k_w(t) \geq 0$.
such that. $0 \leq j \leq k_w(t) \iff wj \in t.$ $[\, k_w(t)$
$= \#$ of children of $w.$



$\pi(223) = 22$

$k_{(22)} = 3$



$\cdot$ g is a Galton–Watson tree: $\begin{cases} P(k_\phi g = n) = p_n, & n \geq 0 \\ P(g^{(i)} \in \cdot \mid k_\phi g = k) = P(g \in \cdot) \end{cases}$

$i = 1, \dots, k$ $\quad g^{(i)}$ = shift tr.

Ref: Le Gall (2005): Random trees and applications. Prob. Surveys. Vol 2. 245-311.

- If $\mathcal{T}$ is a GW tree with offspring law $\{p_k : k=0,1,\dots\}$

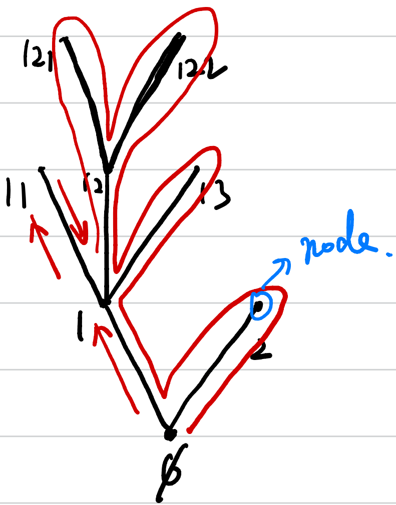
then $Z_n = \#\{w \in \mathcal{T} : |w| = n\}$ $\quad$ $\underbrace{|w|}_{\text{height of } w \text{ in } t.} = k$ if $w = (w_1, \dots, w_k)$

is a GW process.

- For a finite tree t, if $\mathcal{G}$ is a GW tree, then

$$P(\mathcal{G} = t) = \prod_{w \in t} P_{k_w(t)}.$$

④ Some functions Contour function



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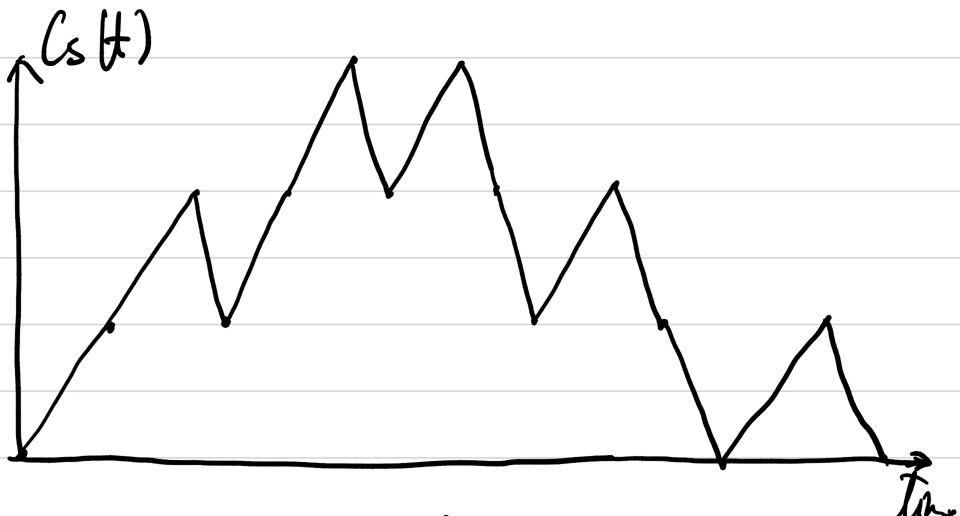
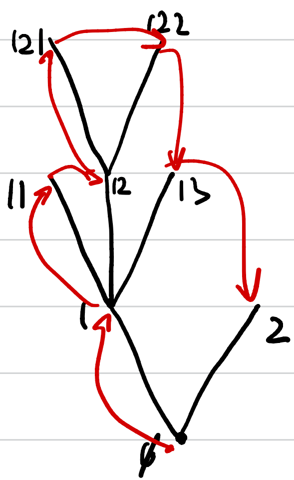


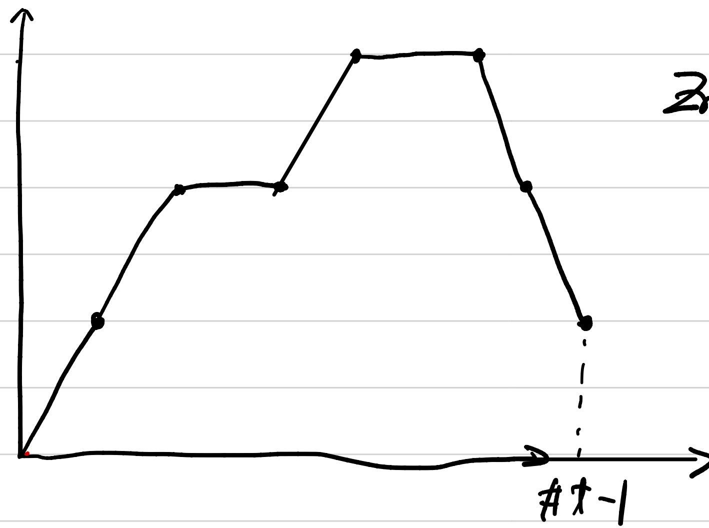
Contour function

$$\{ C_s(t) : 0 \le s \le 2(\#t - 1) \}$$

$\#t$ = # of nodes of t

Height function



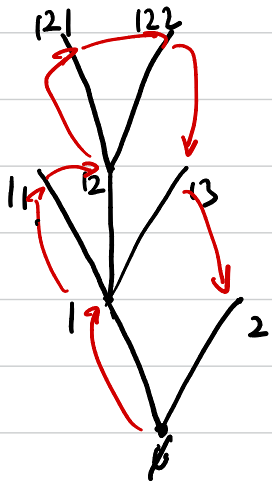


$$Z_n = \#\{s \in \mathbb{Z}^+ : H_s = n\}$$

lexicographical order ↗

Lukasiewicz path.

$t \mapsto (k_{u_0}, k_{u_1}, \cdots, k_{\# t-1}) =: (m_1 \cdots m_{\# t})$. $X_n = \sum_{i=1}^{n} (m_i - 1)$. $X_\phi = 0$



$\vec{m} := (2, 3, 0, 2, 0, 0, 0, 0)$

Prop.: $\nu(k) = p_{k+1}$, $k = -1, 0, 1, 2, \cdots$

$S_0 = 0$, $(S_n : n \geq 0)$ is a RW with step size $\{\nu(k) : k = -1, 0, 1, \cdots\}$

Then L-Path of a GW tree $\overset{d}{=} (S_0, S_1, \cdots, S_T)$

$T := \inf\{n \geq 1, S_n = -1\}$

Prop: $Lt(n) = \#\{j \in \{0, \cdots, n-1\} : x_j = \inf_{j \leq l \leq n} x_l\}$

④ Scaling limits.

$$
\begin{cases}
\text{RW} \Longrightarrow \text{Lévy} \\
\text{GW process} \Longrightarrow \text{CSBP} \\
\text{GW tree} \Longrightarrow \text{Lévy tree}
\end{cases}
$$

I b)

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda r} - 1 + \lambda r)\, \pi(dr), \quad \lambda \geq 0.$$

$\alpha \in \mathbb{R}$, $\beta \geq 0$, $\pi(dr)$ is a σ-finite measure on $(0,+\infty)$ $\int r \wedge r^2 \pi(dr) < \infty$.

ψ- Lévy process : $E^{\psi}[e^{-\lambda X(t)}] = e^{t\psi(\lambda)}$. $t \geq 0$, $\lambda \geq 0$, $X(0) = 0$.

ψ- continuous state branching processes.

$$E[e^{-\lambda Y_t} | Y_0 = y] = e^{-y u_t(\lambda)}, \quad y > 0.$$

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)).$$

(I) Scaling limit of GW processes. $x>0$.

- $\{P_k^{(n)} : k=0,1,2 \cdots\}_{n\geq 1}$. prob measures
- $Y^{(n)} = (Y_k^{(n)})_{k\geq 0}$ be a GW process with $Y_0^{(n)} = [nx]$
- $\nu^{(n)}\left(\frac{k-1}{n}\right) = P_k^{(n)}$, $k\geq 0$, prob. measure on $[-, \infty)$
- $(\gamma_n, n=1,2,\cdots)$: a non-decreasing sequence of positive numbers. $\gamma_n \to +\infty$
- Define. $G^{(n)}(\lambda) = n\cdot\gamma_n [g^{(n)}(e^{-\lambda/\gamma_n}) - e^{-\lambda/\gamma_n}]$.

$$g^{(n)}(s) = \sum_{k=0}^{+\infty} s^k \cdot P_k^{(n)}.$$

A1). $G^{(n)}(\lambda) \to \Psi(\lambda)$. uniformly on bounded interval

A2). $\left(\frac{1}{n} Y^{(n)}_{[\gamma_n t]},\ t \geq 0\right) \xrightarrow{(d)} \left(Y^{x,y}_t,\ t \geq 0\right)$. $n \to \infty$

A3). $\exists$ a prob. measure ν on $[0,\infty)$ s.t.

$$\left(\nu^{(n)}\right)^{*[n\gamma_n]} \to \mu. \quad \textcircled{a} \quad \int e^{-\lambda x} \nu(dx) = e^{\Psi(\lambda)}$$

Prop. If $\underbrace{\int^{\infty} \frac{d\lambda}{\Psi(\lambda)} < \infty}_{\text{Grey}}$ and $\int_{(0,\varepsilon]} \frac{d\lambda}{|\Psi(\lambda)|} = +\infty$. $\forall\, \varepsilon > 0$.

$(\beta > 0$ or $\int_0^1 r\pi(dr) = +\infty$ $\longrightarrow$ Conservative

then (A1), (A2) and (A3) are equivalent.

See Theorem 3.4 in Grimvall (1974 AoP).

② Cvg to Lévy trees / forest.

- $\mathcal{G}^{(n)}$: $[nx]$ GW trees with offspring dist. $p^{(n)}$
- If $\sum k p_k^{(n)} \le 1$, then define contour functions

$$(C^{(n)}(t) : t \ge 0) = \text{concatenating of } (C_1^{(n)}(t), C_2^{(n)}(t), \cdots)$$

Prop. If $\sum k p_k^{(n)} \le 1$ and $\underbrace{\lim_{n\to\infty} g^{(n)}_{[\delta \gamma_n]}(0)^n > 0}_{\text{implies}}$, $\longrightarrow$ Cvg of extinction time

then $\left(\frac{1}{\gamma_n} C^{(n)}(2n\gamma_n t), t \ge 0\right) \xrightarrow{d} \underbrace{H_{\hat{t}}}_{\text{length function}}$.

$\left(\frac{1}{\gamma_n} H^{(n)}(n\gamma_n t), t \ge 0\right) \xrightarrow{d}$

③ Height function of the Lévy tree. [Grey and conservative]

- Recall $\{X(t): t \geq 0\}$ be the Lévy process with

$$E[e^{-\lambda X(t)}] = e^{t\psi(\lambda)}, \quad X(0) = 0.$$

- $\forall\, t \geq 0.$

$$H_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I_{\{X_s < I_t^s + \varepsilon\}}\, ds.$$

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

Recall: $H_t(n) = \#\{ j \in \{0, \cdots, n-1\} : X_j = \inf_{j \leq l \leq n} X_l \}$

- If $\psi(\lambda) = \beta \lambda^2$, then H_t is a reflected BM.

④ Tree and forest.

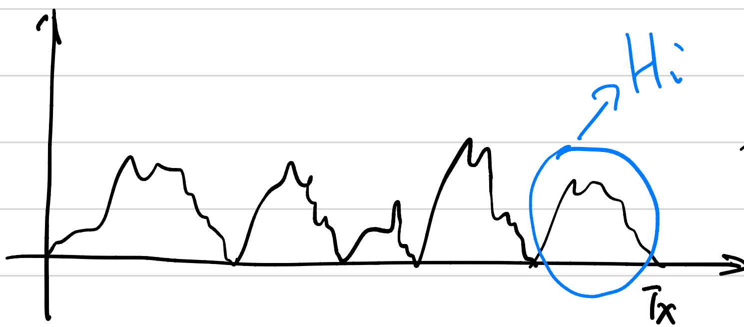
[Bertoin : Lévy processes.
Itô : Poisson process attached to Mk process

$$X_t - \min_{s \le t} X_s$$

- $\underline{X - I}$ = is a strong Markov process with regular point 0.

- $N^Y[\cdot]$: excursion measure of $X-I$. / H. ($N^Y[dH]$) σ-finite measure.

- $\bigcup_{i \in I} (g_i, d_i) = \{ s \ge 0, \ X_s - I_s > 0 \} = \{ s \ge 0 : H_s > 0 \}$.

- 

$\sum \delta_{H_i}(dH)$ is a Poisson Random measure $\sim x \cdot N^Y[dH]$.

⑥ local time and CSBP

$$\lim_{\varepsilon\to 0} E\left[\sup_{s\le t}\left|\frac{1}{\varepsilon}\int_0^s dr \cdot 1_{\{a<H_r\le a+\varepsilon\}} - L_s^a\right|\right]=0$$

and $$\int_0^s g(H_r)\,dr = \int_{\mathbb{R}_+} g(a)\, L_s^a\, da. \quad s\ge 0.$$

$$T_x = \inf\{t\ge 0: I_t \le -x\} = \inf\{s\ge 0: X_s = -x\}$$

Then $\{Z_a = L_{T_x}^a,\ a\ge 0\}$ is a ψ-CSBP starting from x.

· Assume $G_n^{(\lambda)} := n\gamma_n \left[g^{(n)}(e^{-\lambda/\gamma_n}) - e^{-\lambda/\gamma_n} \right] \to \psi(\lambda)$

and $\lim_{n\to\infty} P\left(Y^{(n)}_{[\gamma_n \delta]} = 0 \right) > 0$, $\forall \delta > 0$.

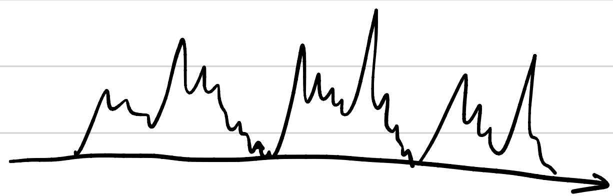
Then $\forall a > 0$,

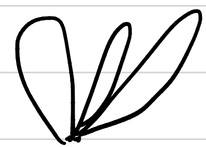
$$P\left[\frac{1}{\gamma_n} H^{(n)}(2n\gamma_n t) \,\middle|\, H(\underset{\downarrow}{\underline{G^{(n)}}}) > [a\gamma_n] \right] \to N^{\psi}\left[dH \mid H(\tau) > a \right]$$

a GW tree [NOT a sequence]

(b) metric space $(\mathcal{T}, d_H, m_{\mathcal{T}})$ → mass measure

= Image measure of Leb. measure under $f \to \mathcal{T}_f$.
mapping

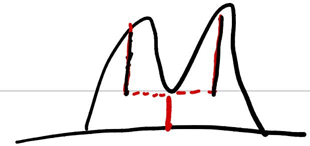




γ - Lévy forest.

$(\mathbb{T}, d_{GH})$ = tree space equipped Gromov-Hausdorff distance.

- $d_f(s,t) = f(s) + f(t) - 2 \inf_{r \in [s,t]} g(r)$



- $\mathcal{T}_f = [0,+\infty) / \sim$ quotient space.

- $(\mathcal{T}_f, d_f)$ is an R-tree.

- $d_{GH}(\mathcal{T}', \mathcal{T}) = \inf \left(d_{Haus}(\varphi(\mathcal{T}), \varphi'(\mathcal{T}')) \vee \delta(\varphi(\rho), \varphi'(\rho')) \right)$

taken over all isometric embeddings

$\varphi : \mathcal{T} \to E$

$\varphi' : \mathcal{T}' \to E$.

(E, δ).

· Assume $G_n^{(\lambda)} := n\gamma_n \left[g^{(n)}(e^{-\lambda/\gamma_n}) - e^{-\lambda/\gamma_n} \right] \to \psi(\lambda)$

and $\displaystyle\lim_{n\to\infty} P\left(Y^{(n)}_{[\gamma_n \delta]} = 0 \right) > 0$, $\forall \delta > 0$.

Then $\forall a > 0$,

$$P\left[\frac{1}{\gamma_n} \mathcal{G}^{(n)} \in \cdot \;\middle|\; H(\mathcal{G}^{(n)}) > [a\gamma_n] \right] \to \underbrace{N^{\psi}[d\mathcal{T}}_{\text{image measure of } N^{\psi}[dH] \text{ under } H \to \mathcal{T}_H} \mid H(\mathcal{T}) > a]$$

7 Degree of branching point

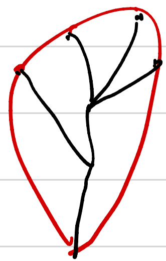
- If $\lambda(du) = 0$, then deg. $= 3$ (only binary branching points)
- If $\lambda(du) > 0$, then deg. $\in \{3, +\infty\}$
- If deg of $v = +\infty$, then $\lim_{\varepsilon \to 0} \frac{n(v,\varepsilon)}{V(\varepsilon)}$ $\exists$ = jumping size of X.
- v is determined by $\int_{v(a)}^{\infty} \frac{du}{\psi(u)} = a$.

- $n(v,\varepsilon) = \#\{T_v : H(T_v) > \varepsilon\}$

⑧ Poissonian sampling

- Let $\sum \delta_{(t_i, x_i)}$ PRM on $[0,+\infty) \times \mathcal{T}_{H}$ $\quad dt \cdot \underbrace{m(dx)}$

  ↳ mass measure.



$$\mathcal{T}_\lambda = \bigcup_{1\{t_i \le \lambda\}} [\phi, x_i] .$$

is a continuous time GW trees under $N^\psi[d\mathcal{T} \mid M(\mathcal{T}) \ge 1]$

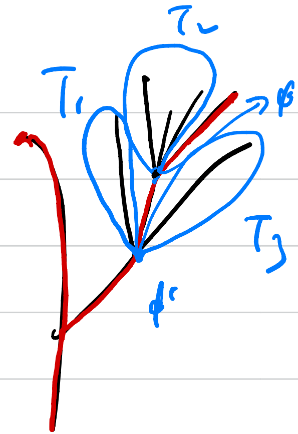
life time $\sim \exp(\psi'(\psi^{-1}(\lambda)))$

offspring law $\sim$ $g_{\psi,\lambda}(r) = r + \dfrac{\psi((1-r)\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)\,\psi'(\psi^{-1}(\lambda))}$.

- Growth process. $\mathcal{T}_\lambda \xrightarrow{\lambda \to +\infty} \mathcal{T}$. a.e.

9 Reconstruction.

· Given $\mathcal{F}_\lambda$.



$T \setminus \mathcal{F}_\lambda = \{T_i^\circ\}$

What is the distribution of $\{T_i\}$ $[T_i = T_i^\circ \cup \{\phi_i'\}]$

Prop. Under $N^\psi[\cdot \mid M_\lambda \geq 1]$, conditionally on $\mathcal{F}_\lambda$.

$\mathcal{T}$ is distributed as

$$\tilde{\mathcal{F}}_\lambda \circledast_{i\in I} (\mathcal{T}_i, x_i) \circledast_{\underbrace{x\in Br(\tilde{\mathcal{F}}_\lambda)}_{\hookrightarrow \text{ branching points of } \tilde{\mathcal{F}}_\lambda.}} (\mathcal{T}'_x, x).$$

- $\tilde{\mathcal{F}}_\lambda \overset{d}{=} \mathcal{F}_\lambda$

- $\sum_{i\in I} \delta_{(x_i, \mathcal{T}_i)}$ is a Poisson point measure on $\tilde{\mathcal{F}}_\lambda \times \mathbb{T}$ with intensity $\underbrace{\ell^{\mathcal{T}_0(x)}(dx)}_{\text{length measure}} \times [2\beta N^{\psi_y}[d\mathcal{T}] + \int_0^\infty r\bar{\pi}(dr)\, P_r^{\psi_y}(d\mathcal{T})]$

$$\psi_y = \psi(\cdot + \psi^{-1}(\lambda)) - \lambda$$

"=" Leb. measure of arc.

- Conditionally on $\sum_{j\in I} \delta_{(x_j, T_j)}$

$(\tilde{T}_x', x \in B_r(\hat{f}_\lambda))$ are independent with $T_x' \overset{d}{=}$

$$\int_0^{+\infty} \Gamma^{\gamma}_{k(x),\lambda}(dr)\, P_r^{x_0}(dT)$$

$$\Gamma^{\gamma}_{d,\lambda}(dr) = I_{\{d=2\}} \frac{2\beta}{\psi''(\eta)} \cdot \delta_0(dr) + \frac{r^d e^{-r\eta}}{[\psi^{(d)}(\eta)]} \pi(dr)$$

$\eta = p^{-1}(\lambda)$.

· Height function

For fixed t, $\quad H_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \int_0^t \mathbb{1}\{X_s \le I_s^t + \varepsilon\}\, ds$

· For $t > 0$, $\quad \hat{X}_s^t = X_{t-} - X_{(t-s)-}, \quad 0 \le s \le t, \qquad \hat{X}_t^t = X_t$

$$\{\hat{X}_s^t, 0 \le s \le t\} \overset{d}{=} \{X_s : 0 \le s \le t\}$$

$$\hat{S}_s^t = \sup_{r \le s} \hat{X}_r^t$$

$s \mapsto t-s$.

$$\{s \in [0,t] : X_{s-} \le \inf_{s \le r \le t} X_r\} \text{ “}=\text{” } \{s \in [0,t] : \hat{S}_s^t = \hat{X}_s^t\}$$

• $H_t = \Gamma_t(\hat{X}^t) \quad t \ge 0$

$\hookrightarrow$ local time at 0 up to time t.