When population models meet Bose-Einstein condensation

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1 Kingman's model

The whole story starts from Kingman's ([7, 1978]) paper on the balance between selection and mutation. Kingman's plan was to introduce a simple model that describes the competition between selection and mutation, and that eventually the population converges to the asymptotic stationary state. A condensation occurs if a non-negligible fraction of the population travels to and condensates at the largest fitness value. Kingman's model exhibits the phase transition phenomenon of condensation, depending on the relative strength of selection and mutation. Bianconi et al [2, 2009] argued that this is akin to the effect of Bose-Einstein condensation, in which for a dilute gas of weakly interacting bosons at very low temperatures a fraction of the bosons occupy the lowest possible quantum state.

More precisely, Kingman considered an infinite population with discrete generations, and the fitness values of an individual are within [0,1]. The population is driven by selection and mutation. The model has three parameters (P_0, Q, b) and is defined as:

$$P_{n+1}(dx) = (1-b)\underbrace{\frac{xP_n(dx)}{\int_0^1 yP_n(dy)}}_{\text{selection}} + b\,\underbrace{Q(dx)}_{\text{mutation}}, \quad n \geq 0. \tag{1.1}$$

- Q, P_n are probability measures on $[0,1]$. Q is the mutant distribution, and P_n is the fitness distribution at the n-th generation for $n \geq 0$.
- $b \in (0,1)$ is deterministic, and is interpreted as the mutation probability for each generation.

Let $h := \sup\left\{x : Q([x,1]) + P_0([x,1]) > 0\right\}$. So h is interpreted as the largest fitness value of the population. Kingman showed that there exists a phase transition depending on $\zeta(b) := 1 - b\int \frac{Q(dy)}{1-y/h}$.

Theorem 1. *If $\zeta(b) \leq 0$, then $(P_n)_{n\geq 0}$ converges strongly to*

$$\frac{b\theta Q(dx)}{\theta - (1-b)x},$$

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with θ being the unique solution of $\int \frac{b\theta Q(dx)}{\theta-(1-b)x} = 1$.

If $\zeta(b) > 0$, then $(P_n)_{n\geq 0}$ converges weakly to

$$\frac{bQ(dx)}{1-x/h} + \zeta(b)\delta_h(dx),$$

here $\delta_h(dx)$ is the Dirac measure at h. Condensation occurs.

Below we list some generalisations of Kingman's model.

Lenski experiment[9, 2017]: Fix $\lambda > 0$. Consider a population model as follows

$$P_{n+1}(dx) = (1-b)\frac{e^{t_n x}P_n(dx)}{\lambda} + bQ(dx), \quad n \geq 0,$$

where t_n is a number such that

$$\int e^{t_n x}P_n(dx) = \lambda.$$

Bürger's model[3, 1988]: Let $r(x,y)$ denote the conditional probability density for mutation from type y to type x. Let $p_n(x)$ denote the density function of the n-th generation. Then

$$p_{n+1}(dx) = (1-b)\frac{w(x)p_n(x)}{\int w(z)p_n(z)dz} + b\int r(x,y)\frac{w(y)p_n(y)}{\int w(z)p_n(z)dz}dy, \ n \geq 0.$$

Continuous-time model[1, 2018]: Let $\mathcal{M}$ be a certain set of nonnegative finite measures on $\mathbb{R}_+$. Let $B : \mathcal{M} \mapsto C(\mathbb{R}_+), C : \mathcal{M} \mapsto C(\mathbb{R}_+)$. Fix $\alpha > 0$ and define

$$\partial_t P_t(dx) = B[P_t]P_t(dx) + x^\alpha C[P_t]dx.$$

To see why it is a generalisation of Kingman's model, note that (1.1) is equivalent to

$$P_{n+1}(dx) - P_n(dx) = \underbrace{(1-b)\left(\frac{x}{\int yP_n(dy)} - 1\right)}_{B[P_n]}P_n(dx) + \underbrace{bQ(dx)}_{x^\alpha C[P_n]dx}$$

2 Peter Mörters et al: condensation wave

If condensation occurs, how fast the mass travels to h, the largest fitness value? Dereich and Mörters showed the following for Kingman's model.

Theorem 2. *WLOG assume that $h = 1$. Suppose that the mutant distribution Q satisfies*

$$\lim_{u\to 0}\frac{Q(1-u,1)}{u^\alpha} = 1,$$

where $\alpha > 1$. If Kingman's model falls in the condensation regime (i.e. $\zeta(b) > 0$), then for any $x > 0$,

$$\lim_{n\to\infty}\mathbb{P}_n\left(1-\frac{x}{n},1\right) = \frac{\zeta(b)}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-y}dy.$$

The Gamma function appears! They made the conjecture that in similar models with a pair of competing forces, when the condensation occurs, the traveling wave should be of the shape of the Gamma function. They worked in a series of models, using various frameworks such as branching processes [5, 2017], preferential attachment networks [4, 2016], zero-range processes [8, 2016], and random permutations [6, 2015].

3 How does extra randomness affect the condensation?

I was interested in the following model. We use the same P_0 and Q from Kingman's model but replace b by a sequence of i.i.d. random variables $\beta_n \in [0,1], n \geq 0$:

$$P_{n+1}(dx) = (1-\beta_n)\frac{xP_n(dx)}{\int yP_n(dy)} + \beta_n Q(dx), \quad n \geq 0.$$

- Does $(P_n)_{n\geq 0}$ converge?
- If so, what is the condensation criterion?
- Is it easier to have condensation in the random model or Kingman's model?
- If we set $\mathbb{E}[\beta_1] = b$, is the limit fitter in the random model than in Kingman's model or the other way round?

These questions were answered in [10, 2020] and [11, 2022]. In particular, it was shown that the extra randomness will hinder the condensation. Is it true in general? A conjecture to explore in various models.

4 Connection to random matrices

4.1 Some facts

Let μ be a probability measure on $[0,\infty)$. Let $f_n = \int_0^\infty x^n \mu(dx)$ for $n \geq 1$. Define a renewal sequence $(u_n)_{n\geq 0}$ as follows

$$u_0 = 1; \quad u_n = \sum_{r=1}^{n} f_r u_{n-r}, n \geq 1.$$

Then there exists a probability measure ν on $[0,\infty)$ such that $u_n = \int_0^\infty x^n \nu(dx), \forall n$. We refer to kingman1978simple for a discussion.

Moreover, it is easy to verify that

$$u_n = det\begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ -1 & f_1 & f_2 & \cdots & f_{n-1} \\ 0 & -1 & f_1 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & f_1 \end{pmatrix}, \quad n \geq 1. \tag{4.2}$$

To my best knowledge, this representation was only given in [11, 2020].

4.2 The random model

Consider a finite backward sequence $(P_j^n)_{0\leq j\leq n}$ of the random Kingman's model:

$$P_n^n = P_0; \quad P_j^n = (1-\beta_{j+1})\frac{xP_{j+1}^n(dx)}{\int yP_{j+1}^n(dy)} + \beta_{j+1}Q(dx), \; 0 \leq j \leq n-1.$$

Define $\gamma_j = \frac{1-\beta_j}{\beta_j}, m_j = \mathbb{E}[(\beta_1)^j],\ \forall j$. If $P_n^n = Q$, then

$$\frac{xP_j^n(dx)}{\int yP_j^n(dy)} = \frac{det\begin{pmatrix} x & x^2 & x^3 & \cdots & x^{n-j+1} \\ -\gamma_{j+1} & m_1 & m_2 & \cdots & m_{n-j} \\ 0 & -\gamma_{j+2} & m_1 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\gamma_n & m_1 \end{pmatrix}}{det\begin{pmatrix} m_1 & m_2 & m_3 & \cdots & m_{n-j+1} \\ -\gamma_{j+1} & m_1 & m_2 & \cdots & m_{n-j} \\ 0 & -\gamma_{j+2} & m_1 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\gamma_n & m_1 \end{pmatrix}} Q(dx).$$

The question is: can we find asymptotic properties of these deterministic and random matrices, such as the eigenvalues? An open problem that seems interesting.

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