An introduction to superprocess: as the scaling Limit of branching particle system

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Overview

1. Scaling limit for branching particle system

2. Definition for superprocess with general (local) branching mechanism

3. Markov property/ Mean formula/ $\mathbb{N}$-measure/ exit measure

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Branching Brownian motion: integral equation

- Branching rate: $\beta > 0$; offspring distribution: $\{p_k :\in \mathbb{N}\}$.
- Z_t : the point process formed by the position of all particles alive at time t, i.e. $Z_t := \sum_{u\in\mathcal{N}_t} \delta_{Z_u(t)}$, where $\mathcal{N}_t$ is the set of particle alive at time t and $Z_u(t)$ is the position of $u \in \mathcal{N}_t$.
- $\mathcal{M}_a(\mathbb{R})$: set of finite point process in $\mathbb{R}$. For any function f and $\mu \in \mathcal{M}_a(\mathbb{R})$, define $\langle f, \mu\rangle := \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x)$.
- For any bounded non-negative Borel function f, set

$$\mathbb{P}_{\delta_x}\left(\exp\{-\langle f, Z_t\rangle\}\right) =: \exp\left\{-u^f(t,x)\right\}.$$

By considering the first splitting time and branching property,

$$\begin{aligned} e^{-u^f(t,x)} &= e^{-\beta t}\Pi_x\left(e^{-f(B_t)}\right) + \int_0^t \beta e^{-\beta s}\Pi_x\left(\sum_{k=0}^{\infty} p_k\left(e^{-u^f(t-s,B_s)}\right)^k\right)\mathrm{d}s \\ &=: e^{-\beta t}\Pi_x\left(e^{-f(B_t)}\right) + \int_0^t \beta e^{-\beta s}\Pi_x\left(g\left(e^{-u^f(t-s,B_s)}\right)\right)\mathrm{d}s \end{aligned}$$

Branching Brownian motion: integral equation

- For $0 < r < t$, replacing (t, x) by $(t-r, B_r)$ yields that

$$e^{-u^f(t-r,B_r)} = e^{-\beta(t-r)}\Pi_{B_r}\left(e^{-f(B_{t-r})}\right) + \int_0^{t-r} \beta e^{-\beta s}\Pi_{B_r}\left(g\left(e^{-u^f(t-r-s,B_s)}\right)\right)\mathrm{d}s.$$

  Note that $1 - e^{-\beta t} = \int_0^t \beta e^{-\beta(t-r)}\mathrm{d}r$, we get the following integral equation:

$$e^{-u^f(t,x)} = \Pi_x\left(e^{-f(B_t)}\right) + \int_0^t \Pi_x\left(\phi\left(e^{-u^f(t-s,B_s)}\right)\right)\mathrm{d}s. \tag{1}$$

  where $\phi(x) := \beta\left(g(x) - x\right)$.
- For a general branching Markov process of which the branching rate and offspring distribution can be spatially dependent, we can also have the similar integral equation as (1).

Scaling limit of branching Brownian motion

- Suppose that there is a sequence of branching particle systems $\left\{\left(Z_t^{(k)}, t \geq 0, \mathbb{P}_\mu^{(k)}, k\mu \in \mathcal{M}_a(\mathbb{R})\right),\ k = 1, 2, \dots\right\}$ such that for every k, $Z_t^{(k)}$ is a branching Brownian motion with $Z_0^{(k)} = k\mu$, branching rate β_k and the generating function of the offspring is given by $g_k(x)$. Let $X_t^{(k)} := k^{-1} Z_t^{(k)}$.
- Define

$$\mathbb{P}_\mu^{(k)}\left(\exp\left\{-\langle f, X_t^{(k)}\rangle\right\}\right) =: \exp\left\{-\langle u_k^f(t,x), \mu\rangle\right\},$$

then by (1), let $\mu = k^{-1}\delta_x$, we see that $u_k^f(t,x)$ solves the equation

$$\begin{aligned} e^{-k^{-1}u_k^f(t,x)} &= \Pi_x\left(e^{-k^{-1}f(B_t)}\right) \\ &+ \int_0^t \Pi_x\left(\beta_k\left(g_k\left(e^{-k^{-1}u_k^f(t-s,B_s)}\right) - k^{-1}u_k^f(t-s,B_s)\right)\right)\mathrm{d}s. \end{aligned}$$

Scaling limit of branching Brownian motion

- Let $v_k^f(t,x) := k\left(1 - e^{-k^{-1}u_k^f(t,x)}\right)$, then $v_k^f(t,x)$ is the unique solution to

$$v_k^f(t,x) = k\Pi_x\left(1 - e^{-k^{-1}f(B_t)}\right) - \int_0^t \Pi_x\left(\phi_k\left(v_k(t-s,B_s)\right)\right)\mathrm{d}s,$$

  where $\phi_k(z) := k\beta_k\left(g_k(1-k^{-1}z) - (1-k^{-1}z)\right)$.
- In Li's[1, Section 4.2] book:
  Condition 4.2: For each $a \geq 0$, $\phi_k(z)$ is Lipschitz with respect to z uniformly on $[0,a]$ and $\phi_k(z) \to \phi(z)$ uniformly on $[0,a]$.
- When ϕ_k is spatially dependent and $\mathbb{R}$ is replaced to a Lusin topological space E, i.e. E is homeomorphic to a Borel subset of a compact metric space. Condition 4.2 is modified as:
  Condition 4.2': For each $a \geq 0$, $\phi_k(x,z)$ is Lipschitz with respect to z uniformly on $E \times [0,a]$ and $\phi_k(x,z) \to \phi(x,z)$ uniformly on $E \times [0,a]$.

Scaling limit of branching Brownian motion.

- (i) [1, Proposition 4.3.]: If ϕ_k satisfies Condition 4.2., then $\phi(z)$ must has representation

$$\phi(z) = \alpha z + \beta z^2 + \int_0^\infty \left(e^{-uz} - 1 + uz\right) \nu(\mathrm{d}u), \tag{2}$$

where $\alpha \in \mathbb{R}, \beta \geq 0$ and ν is the measure supported on $(0, \infty)$ with $\int_0^\infty (u \wedge u^2)\nu(\mathrm{d}u) < \infty$.

- [1, Proposition 4.5.]: Moreover, for each $T > 0$ and non-negative bounded function f, $u_k^f(t, x)$ and $v_k^f(t, x)$ converges uniformly on $[0, T] \times \mathbb{R}$ to the unique locally bounded positive solution $(t, x) \to V_t f(x)$ of the evolution equation

$$V_t f(x) = \Pi_x \left(f(B_t)\right) - \int_0^t \Pi_x \left(\phi(V_{t-s} f(B_s))\right) \mathrm{d}s. \tag{3}$$

- [1, Theorem 4.6.]: Let $\mathcal{M}(\mathbb{R})$ be the set of finite measure on $\mathbb{R}$. For any $\mu \in \mathcal{M}(\mathbb{R})$, let $\mathbb{Q}^{(k)}$ be the law of Poisson random measure (PRM) on $\mathbb{R}$ with intensity $k\mu$, also define $\mathbb{P}^{(k)}_{(\mu)}$ by $\mathbb{P}^{(k)}_{(\mu)}(\cdot) := \int_{\mathcal{M}_a(\mathbb{R})} \mathbb{P}^{(k)}_{k^{-1}\eta}(\cdot)\mathbb{Q}^{(k)}(\mathrm{d}\eta)$, then

$$\mathbb{P}^{(k)}_{(\mu)}\left(\exp\left\{-\langle f, X^{(k)}_t\rangle\right\}\right) = \exp\left\{-\langle v^f_k(t,\cdot), \mu\rangle\right\}$$

  and thus $\left\{X^{(k)}_t, \mathbb{P}^{(k)}_{(\mu)}\right\} \xrightarrow{\text{f.d.d.}} \{X_t, \mathbb{P}_\mu\}$.
  (ii) [1, Proposition 4.4.]: For every ϕ having the form (2), we can find ϕ_k satisfying Condition 4.2.
- When the Brownian motion is generalized to Borel right process (see e.g. [1, Definition A.18.]) in a Luzin topological space (ξ_t, Π_x), also the branching mechanism ϕ_k can be spatially dependent satisfying Condition 4.2.', similar scaling limit can also be obtained.

Overview

Definition

- Suppose that (ξ_t, Π_x) is a Borel right process in a Lusin topological space E.
- Let $\mathcal{M}(E)$ be set of finite measure on E.
- The branching mechanism $\phi(x,z)$ is defined by

$$\phi(x,z) = \alpha(x)z + \beta(x)z^2 + \int_0^\infty \left(e^{-uz} - 1 + uz\right) \nu(x, \mathrm{d}u),$$

  where $\alpha \geq 0$ and β are bounded function on E and $(u \wedge u^2)\nu(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$.
- Suppose that $K = \{K(t) : t \geq 0\}$ is a continuous admissible additive functional of ξ, i.e. (i) $K(t)$ is a continuous additive functional of ξ; (ii) for each t, $\omega \mapsto K(t,\omega)$ is measurable with respect to $\sigma(\xi_s, s \geq 0)$; (iii) $\sup_{x \in E} \Pi_x (K(t)) \to 0, \ t \to 0$.
- A continuous $\mathcal{F}_t$-adapted increasing process $K(t)$ is called a continuous additive functional of ξ if $K(0) = 0$ and for every bounded $\mathcal{F}_t$-stopping time T, it holds that $K(T+t) = K(T) + K(t) \circ \theta_T$.

Definition

Superprocess $\{(X_t)_{t\geq 0};\mathbb{P}_\mu,\mu\in\mathcal{M}(E)\}$ *is an* $\mathcal{M}(E)$*-valued Markov process such that for any non-negative bounded measurable function* f*,*

$$\mathbb{P}_\mu\left(e^{-\langle f,X_t\rangle}\right)=e^{-\langle V_tf,\mu\rangle},\quad t\geq 0, \tag{4}$$

where $(t,x)\mapsto V_tf(x)$ *is the unique locally bounded non-negative solution to*

$$V_tf(x)+\Pi_x\left(\int_0^t\phi\left(V_{t-s}f(\xi_s)\right)K(\mathrm{d}s)\right)=\Pi_x(f(\xi_t)),\quad t\geq 0,x\in E.$$

- [1, Theorem 2.21. below]: X_t is also called Dawson-Watanabe superprocess/ (ξ,K,ϕ)- superprocess. When ξ is BM, $K(t)=t$ and ϕ is homogeneous, i.e. ϕ is independent to x, we call X by super-Brownian motion.
- [1, Section 2.1.]: (4) is called *regular branching perperty*, which is implies branching property: the semigroup Q_t of X_t satisfies $Q_t(\mu_1+\mu_2,\cdot)=Q_t(\mu_1,\cdot)*Q_t(\mu_2,\cdot)$ for all $t\geq 0,\mu_1,\mu_2\in\mathcal{M}(E)$.

Overview

Markov Property

- Let $\mathcal{F}_t := \sigma\left(X_s, s \leq t\right)$, then for every positive bounded $Z \in \sigma\left(X_s : s \geq 0\right)$ and $\mu \in \mathcal{M}(\mathbb{R})$,

$$\mathbb{P}_\mu\left(Z \circ \theta_t \,\middle|\mathcal{F}_t\right) = \mathbb{P}_{X_t}\left(Z\right), \quad \mathbb{P}_\mu, \text{ -a.s..}$$

- By Markov property and the definition of V_t, we see that for every positive bounded function f and $t, s > 0$,

$$\begin{aligned} V_{t+s}f(x) &= -\log \mathbb{P}_{\delta_x}\left(e^{-\langle f, X_{t+s}\rangle}\right) = -\log \mathbb{P}_{\delta_x}\left(\mathbb{P}_{\delta_x}\left(e^{-\langle f, X_{t+s}\rangle}\middle|\mathcal{F}_t\right)\right) \\ &= -\log \mathbb{P}_{\delta_x}\left(e^{-\langle V_s f(\cdot), X_t\rangle}\right) = V_t\left(V_s f\right)(x), \end{aligned}$$

which implies that $V_{t+s} = V_t V_s$. $\{V_t, t \geq 0\}$ is called cumulant semigroup.

Mean formula

In this part, we only consider the case that $K(t) = t$.

- For any non-negative bounded function f,

$$\mathbb{P}_\mu\left(\langle f, X_t\rangle\right) = -\lim_{\theta\downarrow 0}\frac{\partial}{\partial\theta}\mathbb{P}_\mu\left(e^{-\langle\theta f, X_t\rangle}\right) = \langle\lim_{\theta\downarrow 0}\frac{\partial}{\partial\theta}V_t(\theta f), \mu\rangle.$$

- Let $T_t f(x) := \lim_{\theta\downarrow 0}\frac{\partial}{\partial\theta}V_t(\theta f)(x)$, then it is easy to see that $T_t f(x)$ solves equation

$$T_t f(x) + \Pi_x\left(\int_0^t \phi'\left(\xi_s, 0+\right) T_{t-s} f(\xi_s)\mathrm{d}s\right) = \Pi_x(f(\xi_t))$$

with $\phi'(x,z) := \frac{\partial}{\partial z}\phi(x,z)$, which is equivalent to

$$T_t f(x) = \mathbb{P}_{\delta_x}\left(\langle f, X_t\rangle\right) = \Pi_x\left(e^{-\int_0^t \phi'(\xi_s, 0+)\mathrm{d}s} f(\xi_t)\right).$$

Infinitely Divisible

The following statement in this page comes from [1, Section 1.4.].

- For a random measure X on E, we say X is infinitely divisible if for any k, there exist iid random measures $X_1, ..., X_k$ such that $X \overset{\mathrm{d}}{=} X_1 + ... + X_k$.
- By Li[1, Theorem 1.36.], there exists a measure λ on E and a measure L on $\mathcal{M}(E)^\circ := \mathcal{M}(E) \setminus \{\mathbf{0}\}$ such that for any non-negative bounded function f,

$$\mathbb{P}\left(e^{-\langle f, X\rangle}\right) = \exp\left\{-\langle f, \lambda\rangle - \int_{\mathcal{M}(E)^\circ}\left(1 - e^{-\langle f, \eta\rangle}\right) L\left(\mathrm{d}\eta\right)\right\},$$

  here $\int_{\mathcal{M}(E)^\circ}(1 \wedge \langle 1, \eta\rangle)L(\mathrm{d}\eta) < \infty$ and (λ, L) is unique.
- Taking $f = \theta$, we see that $-\log\mathbb{P}\left(X = \mathbf{0}\right) = \lim_{\theta\to\infty}\left(\theta\langle 1, \lambda\rangle + L\left(\mathcal{M}(E)^\circ\right)\right)$. Thus, if $\mathbb{P}\left(X = \mathbf{0}\right) > 0$, then $\lambda = \mathbf{0}$ and $L(\mathcal{M}(E)^\circ) < \infty$.

Entrance law

- By branching property, for each $x \in E$ and t, X_t under $\mathbb{P}_{\delta_x}$ is infinitely divisible.
- Assume that $\mathbb{P}_{\delta_x}(X_t = \mathbf{0}) > 0$, then for every $x \in E, t > 0$,

$$e^{-V_t f(x)} = \mathbb{P}_{\delta_x}\left(e^{-\langle f, X_t\rangle}\right) = \exp\left\{-\int_{\mathcal{M}(E)^\circ}\left(1 - e^{-\langle f,\eta\rangle}\right) L_t(x, \mathrm{d}\eta)\right\},$$

  where $L_t(x, \mathcal{M}(E)^\circ) < \infty$.
- Define $Q_t^\circ := Q_t\big|_{\mathcal{M}(E)^\circ}$. Then by $V_{t+s} = V_t V_s$, we get that

$$L_{t+s}(x, \cdot) = \int_{\mathcal{M}(E)^\circ} L_s(x, \mathrm{d}\mu)\, Q_t^\circ(\mu, \cdot). \tag{5}$$

  We say that $\{L_t(x, \cdot), t > 0\}$ is the entrance law for Q_t° if (5) is satisfied for any $s, t > 0$.

$\mathbb{N}$-measure for superprocess

- Li[1, Theorem A.41.]: Let
$\mathbb{D} := \{w = (w_t)_{t\geq 0} : \ w$ is an $\mathcal{M}(E)$-valued càdlàg function on $[0,\infty)$ $\}$.
Suppose that $\mathbb{P}_{\delta_x}(X_t = \mathbf{0}) > 0$ for every $t > 0$ and $x \in E$, then under $\mathbb{P}_{\delta_x}$, there exists a unique σ-finite measure $\mathbb{N}_x$ supported on $\mathbb{W}_0^+ \subset \mathbb{D}$ such that for all $0 < t_1 < ... < t_n$ and all $\eta_1, ..., \eta_n \in \mathcal{M}(E)^\circ$,

$$\begin{aligned}&\mathbb{N}_x\left(w_{t_j} \in \mathrm{d}\eta_j, j = 1, ..., n\right)\\ &\quad = L_{t_1}(x, \mathrm{d}\eta_1)\, Q^\circ_{t_2-t_1}(\eta_1, \mathrm{d}\eta_2) ... Q^\circ_{t_n-t_{n-1}}(\eta_{n-1}, \mathrm{d}\eta_n).\end{aligned} \tag{6}$$

Here for any $w \in \mathbb{W}_0^+$, w is a càdlàg function on $\mathcal{M}(E)$, $w_t = \mathbf{0}$ when $t \neq (0, \zeta(w))$ for some $\zeta(w) > 0$ and $w_t \neq 0$ when $t \notin (0, \zeta(w))$.

- $\{\mathbb{N}_x : x \in E\}$ is also called Kuznetsov Measure/excursion measure. See also Dynkin and Kuznetsov [2, Theorem 1.1.].
- For the homogeneous case and $K(t) = t$, if Grey's condition is fulfilled, i.e. $\int^\infty \phi(z)^{-1}\mathrm{d}z < \infty$ and $\phi(\infty) = \infty$, then $\mathbb{P}_{\delta_x}(X_t = \mathbf{0}) > 0$ for every $t > 0$.
- We see that $\mathbb{N}_x\left(1 - e^{-\langle f, w_t\rangle}\right) = -\log \mathbb{P}_{\delta_x}\left(e^{-\langle f, X_t\rangle}\right) = V_t f(x)$.

Entrance rule

- If we do not have the assumption $\mathbb{P}_{\delta_x}(X_t = \mathbf{0}) > 0$, then we can only get that

$$V_t f(x) = \int_E f(y)\lambda_t(x, \mathrm{d}y) + \int_{\mathcal{M}(E)^\circ} \left(1 - e^{-\langle f, \eta \rangle}\right) L_t(x, \mathrm{d}\eta).$$

  Therefore, it only holds that $L_{t+s}(x, \cdot) \geq \int_{\mathcal{M}(E)^\circ} L_s(x, \mathrm{d}\mu) Q_t^\circ(\mu, \cdot)$, in which case $L_t(x, \cdot)$ is called entrance rule.
- By Li[1, Theorem A.41.], in this case, there still exists a family of σ-finite measure $\{\mathbb{N}_x, x \in E\}$ but on a larger space $\mathbb{W}^+$ such that (6) holds. Here for any $w \in \mathbb{W}^+$, w is a càdlàg function on $\mathcal{M}(E)$ and $w_t \neq \mathbf{0}$ when $t \in (\alpha(w), \zeta(w))$ for some $\alpha(w) < \zeta(w)$ and $w_t = \mathbf{0}$ when $t \notin (\alpha(w), \zeta(w))$. It will not always hold that $\alpha(w) = 0$. Also, we could only have that $\mathbb{N}_x\left(1 - e^{-\langle f, w_t \rangle}\right) \leq V_t f(x)$.

Branching exit Markov system

- Consider a BBM starting from a single particle at site x at time r, whose law is denoted by $\mathbb{P}_{r,x}$. Also, let $Q \subset [0,\infty) \times \mathbb{R}$ be an open subset such that $(r,x) \in Q$.
- The exit measure on Q is defined by

$$X_Q := \sum_{i=1}^{n} \delta_{(t_i,y_i)},$$

where (t_i, y_i) is the first time-space position for the BBM hitting Q^c.

- When $Q = Q_t = [0,t) \times \mathbb{R}$ and $r = 0$, it is clear that $(X_{Q_t}, \mathbb{P}_{0,x})$ is a BBM.
- A similar integral equation for $e^{-\langle f, X_Q \rangle}$ can be obtained and similar idea for the scaling limit from branching Markov process to superprocess, we can define the exit measure for superprocess. More details for the proof can be found in Dynkin[3, 4].

Exit measure for superprocess

- Let $S := [0, \infty) \times E$. Define $\mathbb{O}$ by the class of open subsets of S.
- Dynkin[3, (1.8), (1.9) and Theorem 1.1]: Let $\mathcal{H}$ be the class of bounded non-negative measurable function f in S whose support is in $[0, a) \times E$ for some $a > 0$. Assume that $K(t) = \int_0^t k(s, \xi_s)\mathrm{d}s$ and some other weak conditions on ξ, K and ϕ.
- There exists a family of random measures $(X_Q, \mathbb{P}_\mu : Q \in \mathbb{O}, \mu \in \mathcal{M}(S))$ such that for every $\mu \in \mathcal{M}(S)$ and $f \in \mathcal{H}, , \mathbb{P}_\mu\left(e^{-\langle f, X_Q\rangle}\right) = e^{-\langle V_f^Q, \mu\rangle}$, where $V_f^Q(s, x)$ is the unique positive solution of the equation

$$V_f^Q(s, x) + \Pi_{s,x} \int_s^\tau \phi\left(\xi_r, V_f^Q(r, \xi_r)\right) K(\mathrm{d}r) = \Pi_{s,x} f(\tau, \xi_\tau),$$

  with $\tau := \inf\{r : (r, \xi_r) \notin Q\}$.
- X_Q is unique in the sense of distribution.

Exit measure for super-Brownian motion

- Special Markov property (Dynkin [3, Theorem 1.3.]): let $\mathcal{F}_Q$ be the σ-field generated by $X_{Q'}$, $Q' \subset Q$, and $\mathcal{F}^Q$ be the σ-field generated by $X_{Q''}$, $Q \subset Q''$, then for every positive bounded random variable $Z \in \mathcal{F}^Q$ and $\mu \in \mathcal{M}(S)$, it holds that

$$\mathbb{P}_\mu \left(Z \,\middle|\, \mathcal{F}_Q \right) = \mathbb{P}_{X_Q} (Z) , \quad \mathbb{P}_\mu - \text{a.s.}.$$

- Mean formula for exit measure (see Dynkin [3, (1.20)]): for every positive bounded measurable function f in S, $\mu \in \mathcal{M}(S)$ and every $Q \in \mathbb{O}$,

$$\mathbb{P}_\mu \left(\langle f, X_Q \rangle \right) = \int_S \Pi_{r,x} \left(e^{-\int_r^\tau \phi'(\xi_s, 0+) K(\mathrm{d}s)} f(\tau, \xi_\tau) \right) \mu(\mathrm{d}r \, \mathrm{d}x),$$

where $\tau := \inf\{t : \ (t, \xi_t) \notin Q\}$.

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