Local convergence analysis of augmented Lagrangian method for nonlinear semidefinite programming

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中国科学院
CHINESE ACADEMY OF SCIENCES

Acknowledgements

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♣ S.W. Wang and D., *Local convergence analysis of augmented Lagrangian method for nonlinear semidefinite programming*, arXiv: 2110.10594, 2021.

Outline

Augmented Lagrangian method

Convergence analysis for ALM

Main results

Augmented Lagrangian method

Augmented Lagrangian function

Consider

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & \Phi(x) \\ \text{s.t.} \quad & h(x) = 0 \quad \leftarrow \quad y \end{aligned}$$

Augmented Lagrangian function

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$$\min_{x \in \mathbb{X}} \quad \Phi(x)$$
$$\text{s.t.} \quad h(x) = 0 \quad \leftarrow \quad y$$

Augmented Lagrangian function[1]:

$$L_\sigma(x; y) := \Phi(x) + \langle y, h(x) \rangle + \frac{\sigma}{2} \|h(x)\|^2$$

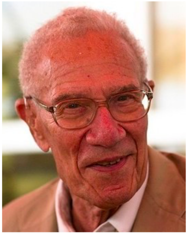
where $\sigma > 0$

[1] Arrow, K.J., Solow, R.M.: Gradient methods for constrained maxima with weakened assumptions. In: Arrow, K.J., Hurwicz, L., Uzawa, H., (eds.) Studies in Linear and Nonlinear Programming. Stanford University Press, Stanford, pp. 165-176 (1958)

K. Arrow and R. Solow

Kenneth Joseph “Ken” Arrow
(23 August 1921 – 21 February 2017)

John Bates Clark Medal (1957); Nobel Prize in Economics (1972); von Neumann Theory Prize (1986); National Medal of Science (2004); ForMemRS (2006)



Robert Merton Solow
(August 23, 1924 –)

John Bates Clark Medal (1961); Nobel Memorial Prize in Economic Sciences (1987); National Medal of Science (1999); Presidential Medal of Freedom (2014); ForMemRS (2006)

Augmented Lagrangian method[2]

ALM (**Hestenes, 69'**; **Powell, 69'**):

$$\begin{cases} x^{k+1} \approx \operatorname{argmin}\left\{ L_\sigma(x; y^k) \right\} \\ y^{k+1} = y^k + \sigma h(x^{k+1}) \end{cases}$$

[2]a.k.a. the method of multipliers

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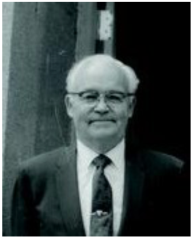


Magnus Rudolph Hestenes

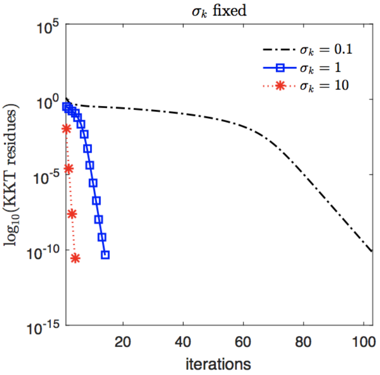
(February 13 1906 – May 31 1991)

Michael James David Powell

(29 July 1936 – 19 April 2015)

[2] a.k.a. the method of multipliers

In particular, for SDP:



Source: Cui, Sun and Toh, MP 19'[3]

[3] Y. Cui, D.F. Sun, K.C. Toh, "On the R-superlinear convergence of the KKT residuals generated by the augmented Lagrangian method for convex composite conic programming", Mathematical Programming 178 (2019) 381–415

ALMs in practice (cont'd)

Roughly speaking, $x^k \to x^\infty$ **linearly** with a rate bounded from above by

$$\frac{\kappa_p}{\sqrt{\kappa_p^2 + \sigma_\infty^2}} < 1 \quad \textbf{(fast linear)}$$

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Powell, 69'[4]: **ALM $\Longleftrightarrow$ Approximate Newton method**

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Newton method for $\Psi(y) := \phi(x(y,\sigma)) = 0$:

$$y \leftarrow y - J\Psi(y)^{-1}\Psi(y) = y - J\Psi(y)^{-1}\phi(x(y,\sigma))$$

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Powell, 69' shows that the Jacobian of Ψ satisfies

$$\|-J\Psi - I\| = O(\frac{1}{\sigma - c})$$

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Convergence analysis for ALM

Local convergence rate of ALM (I)

Convex OPs:

- **Rockafellar, 76'**:

  Dual upper Lipschitz continuity + dual boundedness + stopping criteria

  $\Longrightarrow$ dual Q-linear

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- Cui, Sun and Toh, **MP** 19':

  Dual calmness + stopping criteria

  $\Longrightarrow$ dual Q-(asymptotically super-)linear

  + KKT residual R-(asymptotically super-)linear

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Key fact: the strong connection with dual PPA

Local convergence rate of ALM (II)

NLP (non-convex):

- cf. e.g., Bertsekas, 82'; Nocedal and Wright, 06':

  SOSC + **LICQ** + **strict complementarity** $\Longrightarrow$ primal R-linear+dual Q-linear

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- Fernández and Solodov, **SIOPT** 12':

  SOSC $+$ **initial multiplier sufficiently close** $\Longrightarrow$ primal-dual linear

  Key fact: For NLP, automatically,

  SOSC $\Longrightarrow$ a primal-dual error bound (Hoffman's error bound)

Local convergence rate of ALMs (III)

Non-polyhedral & non-convex:

- NLSDP (Sun, Sun and Zhang, **MP** 08'):

  strong SOSC + **LICQ** + initial sufficiently close $\Longrightarrow$ primal-dual linear

Local convergence rate of ALMs (III)

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- C^2-cone reducible problems (Kanzow and Steck, **MP** 18'):

  $$\text{SOSC} + \textbf{SRCQ} \quad \Longrightarrow \quad \text{primal-dual linear}$$

  Key fact: (**D.**, Sun and Zhang, **SIOPT** 17')

  $$\text{SOSC} + \text{SRCQ} \iff \text{robust isolated calmness}$$

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  SOSC + SRCQ $\Longleftrightarrow$ robust isolated calmness

For **non-convex** and **non-polyhedral** cases (MOPs),

- all results are obtained under the solution **uniqueness** assumption
- unlike the polyhedral case (NLPs), Hoffman's error bound does not hold in general

Local convergence rate of ALM (IV)

- Piecewise linear quadratic (Hang & Sarabi, **SIOPT** 21'):

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- Second order cone (SOC) (Hang, Mordukhovich and Sarabi, **J. Global Opt.** 21'):

  SOSC + **semi-isolated calm** + initial sufficiently close
  + **multiplier unique** $\Longrightarrow$ primal-dual linear

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- Fully amenable (Rockafellar, **MP** 21'): Piecewise linear quadratic & SOC

  strong variational sufficiency $\Longrightarrow$ primal R-linear + dual Q-linear

Main results

Local convergence analysis of ALM for NLSDP

NLSDP:

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & h(x) = 0, & \leftarrow \quad y \\ & G(x) \in \mathcal{S}^n_+, & \leftarrow \quad \Gamma \end{array}$$

Let $\lambda = (y, \Gamma)$, $\Phi(x) = (h(x), G(x))$ and $\mathcal{K} = \{0\} \times \mathcal{S}^n_+$.

Augmented Lagrangian function:

$$\mathcal{L}(x, \lambda, \rho) := f(x) + \frac{\rho}{2}\text{dist}^2(\Phi(x) + \frac{\lambda}{\rho}, \mathcal{K}) - \frac{\|\lambda\|^2}{2\rho}$$

KKT:

$$S_{KKT}(a_1, a_2, b) = \left\{ (x, y, \Gamma) \in \mathcal{X} \times \Re^e \times \mathcal{S}^n : \begin{array}{l} \nabla_x L(x, y, \Gamma) - a_1 = 0, \\ h(x) - a_2 = 0, \\ 0 \preceq (G(x) - b) \perp \Gamma \preceq 0. \end{array} \right\}$$

For a stationary point $\bar{x}$, the set of multipliers satisfying KKT system:

$$\mathcal{M}(\bar{x}) = \{(y, \Gamma) \in \Re^e \times \mathcal{S}^n \mid (\bar{x}, y, \Gamma) \in S_{\text{KKT}}(0, 0, 0)\}.$$

Algorithm 1

Input: Let $(x^0, \lambda^0) \in \mathcal{X} \times \mathcal{H}$, $\rho_0 > 0$, $\varsigma > 1$, $\xi \in (0,1)$, $\{\epsilon_k\}_{k \geq 0}$ with $\epsilon_k > 0$ for all k and $\epsilon_k \to 0$ and set $k := 0$.

Output: x, λ

1: If (x^k, λ^k) satisfies a suitable termination criterion: STOP.

2: Compute $x^{k+1} \approx \mathrm{argmin}\{\mathcal{L}(x, \lambda^k, \rho^k)\}$ such that $\|\nabla_x \mathcal{L}(\cdot, \lambda^k, \rho^k)\| \leq \epsilon_k$.

3: Update the vector of multipliers to

$$\lambda^{k+1} := \rho^k \big[\Phi(x^{k+1}) + \frac{\lambda^k}{\rho^k} - \Pi_{\mathcal{K}}(\Phi(x^{k+1}) + \frac{\lambda^k}{\rho^k})\big].$$

Update ρ^{k+1} by $\rho^{k+1} = \rho^k$ or $\rho^{k+1} = \varsigma\rho^k$ according to certain rules.

4: Set $k \leftarrow k + 1$ and go to **Step 1**.

An assumption

Assumption

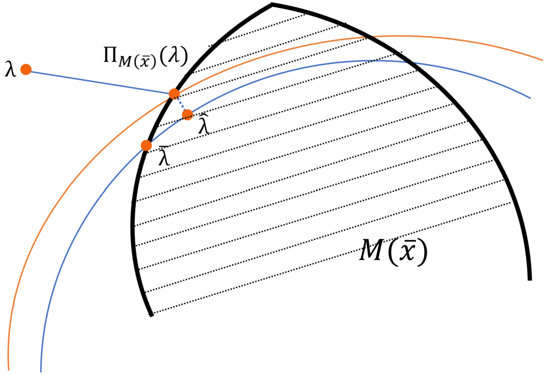
For all $\lambda = (y, \Gamma) \notin \mathcal{M}(\bar{x})$ *sufficiently close to* $\bar{\lambda} \in \operatorname{rbd} \mathcal{M}(\bar{x})$ *and* x *sufficiently close to* $\bar{x}$*, there also exists* $\widehat{\lambda} \in \mathcal{M}(\bar{x})$ *with* $\pi(\widehat{\Gamma}) = \pi(\overline{\Gamma})$ *such that*

$$\|\Pi_{\mathcal{M}(\bar{x})}(\lambda) - \widehat{\lambda}\| = O(R(x, \lambda)),$$

where $R(x, \lambda)$ *is the residual function defined by*

$$R(x, \lambda) = \|\nabla_x L(x, \lambda)\| + \|\Phi(x) - \Pi_{\mathcal{K}}(\Phi(x) + \lambda)\|.$$

An assumption (cont'd)



Main result

Theorem

Let $\bar{x} \in \mathcal{X}$ be a stationary point to NLSDP and $\bar{\lambda} \in \mathcal{M}(\bar{x})$. Suppose SOSC holds at $(\bar{x}, \bar{\lambda})$ and semi-isolated calmness of S_{KKT} holds at $(0, (\bar{x}, \bar{\lambda}))$.

(i) *If $\bar{\lambda} \in \mathrm{ri}\, \mathcal{M}(\bar{x})$, then there exist positive constants $\overline{r}$, $\overline{\zeta}$, $\overline{\varrho}$ such that for any starting point $(x^0, \lambda^0) \in \mathbb{B}_{\overline{r}}(\bar{x}, \bar{\lambda})$ the $\{(x^k, \lambda^k)\}_{k \geq 0}$ generated by ALM with $\rho^k \geq \overline{\varrho}$ and $\epsilon_k = o(R(x^k, \lambda^k))$ for all k satisfies*

$$\|x^{k+1} - x^k\| + \|\lambda^{k+1} - \lambda^k\| \leq \overline{\zeta} R(x^k, \lambda^k).$$

(ii) *If $\bar{\lambda} \in \mathrm{rbd}\, \mathcal{M}(\bar{x})$ and Assumption holds, then there exist positive constants $\overline{r}$, $\overline{\zeta}$, $\overline{\varrho}$ such that for any starting point $(x^0, \lambda^0) \in \mathbb{B}_{\overline{r}}(\bar{x}, \bar{\lambda})$ the $\{(x^k, \lambda^k)\}_{k \geq 0}$ generated by ALM with $\rho^k \geq \overline{\varrho}$ and $\epsilon_k = o(R(x^k, \lambda^k))$ and $\lambda^k \notin \mathcal{M}(\bar{x})$ for all k also satisfies the above inequality.*

Moreover, for each case, the sequence is convergent to $(\bar{x}, \widehat{\lambda})$ for some $\widehat{\lambda} \in \mathcal{M}(\bar{x})$ and its rate of convergence is linear, i.e., for k sufficiently large,

$$\|(x^{k+1}, \lambda^{k+1}) - (\bar{x}, \widehat{\lambda})\| \leq \tau^k \|(x^k, \lambda^k) - (\bar{x}, \widehat{\lambda})\|,$$

where $\tau^k = 2\sqrt{2\zeta}\kappa_1 \kappa_2^2 (R_k^{-1} \epsilon_k + (\rho^k)^{-1} \overline{\zeta})$.

Spectral operators on matrices

Spectral operators: a class of matrix-valued functions defined on the spectral. (**D.** PhD thesis; **D.**, et. al, **MP** 18' and **SIOPT** 20')

For instance, the metric projection operator $\Pi_{\mathcal{S}^n_+}(\cdot) : \mathcal{S}^n \to \mathcal{S}^n$ over $\mathcal{S}^n_+$:

$$\Pi_{\mathcal{S}^n_+}(X) = P \begin{bmatrix} (\lambda_1(X))_+ & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & (\lambda_n(X))_+ \end{bmatrix} P^T, \quad X \in \mathcal{S}^n,$$

where $\lambda_1(X) \geq \ldots \geq \lambda_n(X)$ are eigenvalues and P is the corresponding eigenvector matrix, i.e.,

$$X = P \begin{bmatrix} \lambda_1(X) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \lambda_n(X) \end{bmatrix} P^T.$$

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Let $\pi(X) := \cup_{i=1}^d \alpha^l(X)$ be the partition of eigenvalues $\lambda(X)$ with $\alpha^l(X) := \{i : \lambda_i(X) = v_i(X)\}$, where $v_1(X) > \cdots > v_d(X)$ are different eigenvalues of X.

A refined perturbation analysis of eigenvalue decompositions

Given a fixed $\overline{A} \in \mathcal{S}^n$. Let $0 < r < \min_{i<j}\{v_i(\overline{A}) - v_j(\overline{A})\}/3$. For any $H \in \mathcal{S}^n$ and $A \in \mathbb{B}_r(\overline{A})$, let U be an orthogonal matrix such that

$$U^T\big(\Lambda(A) + H\big)U = \Lambda\big(\Lambda(A) + H\big).$$

Lemma

Then, for any $H \to 0$, we have

$$\begin{cases} U_{\bar{\alpha}_k\bar{\alpha}_l} = O(\|H\|), & k, l = 1, \cdots, \bar{d}, k \neq l \\ U_{\bar{\alpha}_k\bar{\alpha}_k}U^T_{\bar{\alpha}_k\bar{\alpha}_k} = I_{|\bar{\alpha}_k|} + O(\|H\|^2), & k = 1, \cdots, \bar{d} \end{cases}$$

Furthermore, for each $k \in \{1, \ldots, \bar{d}\}$, there exists $Q_k \in \mathcal{O}^{|\bar{\alpha}_k|}$ such that

$$U_{\bar{\alpha}_k\bar{\alpha}_k} = Q_k + O(\|H\|^2)$$

and

$$Q_k^T H_{\bar{\alpha}_k\bar{\alpha}_k} Q_k = \Lambda_{\bar{\alpha}_k\bar{\alpha}_k}(\Lambda(X) + H) - Q_k^T \Lambda(A)_{\bar{\alpha}_k\bar{\alpha}_k} Q_k + O(\|H\|^2).$$

The $O(\|H\|)$ and $O(\|H\|^2)$ above are uniform for all $A \in \mathbb{B}_r(\overline{A})$.

Given $\overline{A} \in \mathcal{S}^n$ and let $0 < r < \min_{i<j}\{v_i(\overline{A}) - v_j(\overline{A})\}/3$. For any $H \in \mathcal{S}^n$ and $A \in \mathbb{B}_r(\overline{A})$, let U be an orthogonal matrix such that

$$U^T\big(A + H\big)U = \Lambda(A + H).$$

Lemma

For all $l \in \{1, \ldots, \bar{d}\}$, there exist $Q_l \in \mathcal{O}^{|\bar{\alpha}^l|}$ (depends on H) such that for all $H \to 0$,

$$(P^T U)_{\bar{\alpha}_k \bar{\alpha}_l} = \Theta_{kl} \circ (\widetilde{H}_{\bar{\alpha}_k \bar{\alpha}_l} Q_l) + O(\|H\|^2), \quad k \neq l$$

where $O(\|H\|^2)$ is uniform for all $A \in \mathbb{B}_r(\overline{A})$,
$(\Theta_{kl})_{ij} = 1/\big((\Lambda(A)_{\bar{\alpha}_l \bar{\alpha}_l})_{ii} - (\Lambda(A)_{\bar{\alpha}_k \bar{\alpha}_k})_{jj}\big)$ and $\widetilde{H} = P^T H P$, $P \in \mathcal{O}^n(A)$.

Proposition

Let $\overline{A} \in \mathcal{S}^n$ be given. The metric projection operator $\Pi_{\mathbb{S}^n_+}(\cdot)$ over $\mathbb{S}^n_+$ is **uniformly** *B-differentiable of order 2 for any $A \in \mathbb{B}_r(\overline{A})$ with $\pi(\overline{A}) = \pi(A)$, i.e., for $\mathcal{S}^n \ni H \to 0$,*

$$\Pi_{\mathcal{S}^n_+}(A+H) - \Pi_{\mathcal{S}^n_+}(A) - \Pi'_{\mathcal{S}^n_+}(A;H) = O\left(\|H\|^2\right)$$

and $O\left(\|H\|^2\right)$ is uniform for all $A \in \mathbb{B}_r(\overline{A})$ with $\pi(\overline{A}) = \pi(A)$.

$\star$ In literature, we only know that $\Pi_{\mathcal{S}^n_+}(\cdot)$ is B-differentiable of order 2 (**D.**, et al, **MP** 14').

Uniformly second order expansion for Moreau envelop of SDP

Proposition

Let $\overline{A} \in \mathcal{S}^n$ be given. For any $A \in \mathbb{B}_r(\overline{A})$ with $\pi(A) = \pi(\overline{A})$, we have for all $H \to 0$,

$$e\delta_{\mathcal{S}^n_+}(A+H) - e\delta_{\mathcal{S}^n_+}(A) = \langle \Pi_{\mathcal{S}^n_-}(A), H\rangle + \frac{1}{2} e\big(\mathrm{d}^2\delta_{\mathcal{S}^n_+}(G(\bar{x}),\Gamma)\big)(H) + O(\|H\|^3),$$

where $O(\|H\|^3)$ is **uniform** *for all $A \in \mathbb{B}_r(\overline{A})$ with $\pi(A) = \pi(\overline{A})$ and $\mathrm{d}^2\delta_{\mathcal{S}^n_+}(G(\bar{x}),\Gamma)$.*

- $\pi(A) = \pi(\overline{A})$ means $\alpha_l(A) = \alpha_l(\overline{A})$ for all $k = 1, \ldots, d$
- (Poliquin & Rockafellar , **SIOPT** 96') generated the non-uniform version of $o(\|H\|^2)$

Uniform quadratic growth condition for AL function

Theorem

Let $\bar{x} \in \mathcal{X}$ be a locally optimal solution to the NLSDP and $\bar{\lambda} \in \mathcal{M}(\bar{x})$.

(i) *If $\bar{\lambda} \in \mathrm{ri}\, \mathcal{M}(\bar{x})$, then the SOSC holds at $(\bar{x}, \bar{\lambda})$ if and only if there are positive constants ρ_3, γ, ε, l such that for all $\lambda \in \mathcal{M}(\bar{x}) \cap \mathbb{B}_\varepsilon(\bar{\lambda})$ and all $\rho \geq \rho_3$ the uniform quadratic growth condition*

$$\mathcal{L}(x, \lambda, \rho) \geq f(\bar{x}) + l\|x - \bar{x}\|^2 \quad \text{for all } x \in \mathbb{B}_\gamma(\bar{x}) \tag{1}$$

is satisfied.

(ii) *If $\bar{\lambda} \in \mathrm{rbd}\, \mathcal{M}(\bar{x})$, then the SOSC holds at $(\bar{x}, \bar{\lambda})$ if and only if there are positive constants ρ_3, γ, ε, l such that (1) holds uniformly for all $\rho \geq \rho_3$ and all $\lambda \in \mathcal{M}(\bar{x}) \cap \mathbb{B}_\varepsilon(\bar{\lambda})$ with $\pi(\Gamma) = \pi(\overline{\Gamma})$.*

Uniform quadratic growth condition for AL function

Theorem

Let $\bar{x} \in \mathcal{X}$ be a locally optimal solution to the NLSDP and $\bar{\lambda} \in \mathcal{M}(\bar{x})$.

(i) *If $\bar{\lambda} \in \operatorname{ri} \mathcal{M}(\bar{x})$, then the SOSC holds at $(\bar{x}, \bar{\lambda})$ if and only if there are positive constants ρ_3, γ, ε, l such that for all $\lambda \in \mathcal{M}(\bar{x}) \cap \mathbb{B}_\varepsilon(\bar{\lambda})$ and all $\rho \geq \rho_3$ the uniform quadratic growth condition*

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- While in polyhedral case, uniform quadratic growth condition can be get directly from SOSC

Comparison: Rockafellar's variational sufficiency

Consider a general parametrical optimization

$$\min \varphi(x,u) \quad \text{s.t. } u = 0,$$

where u is a perturbation parameter. Let $\varphi_\rho(x,u) = \varphi(x,u) + \frac{\rho}{2}|u|^2$.

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Definition (Rockafellar, MP 21' [5])

The **(strongly) variational sufficiency** for local optimality holds with respect to $(\bar{x},\bar{y}) \in \operatorname{gph}\partial\varphi_\rho$ satisfying the first-order condition if there exists $\rho > 0$ such that is **variationally (strongly) convex** with respect to $((\bar{x},0),(0,\bar{y}))$ in $\operatorname{gph}\partial\varphi_\rho$, i.e., there exist open convex neighborhoods $\mathcal{W}$ of $(\bar{x},0)$ and $\mathcal{Z}$ of $(0,\bar{y})$ such that

there exists a proper lsc **(strongly) convex function** $\psi \le \varphi_\rho$ on $\mathcal{W}$ such that

$$[\mathcal{W}\times\mathcal{Z}) \cap \operatorname{gph}\partial\psi = [\mathcal{W}\times\mathcal{Z}) \cap \operatorname{gph}\partial\varphi_\rho$$

and, for $(x,u;v,y)$ belonging to this common set, $\psi(x,u) = \varphi_\rho(x,u)$.

[5] R.T. Rockafellar. Augmented Lagrangians and hidden convexity in sufficient conditions for local optimality. in *Math. Programming*, to appear.

Comparison: Rockafellar's variational sufficiency (cont'd)

Theorem (Rockafellar, MP 21')

With respect to $\bar{x}$ *and* $\bar{y}$ *satisfying the first-order optimality condition, the* **variational sufficiency** *for local optimality holds if and only if, for* $\rho > 0$ *sufficiently large, there is a closed convex neighborhood* $\mathcal{X} \times \mathcal{Y}$ *of* $(\bar{x}, \bar{y})$ *such that* $\mathcal{L}(x, y, \rho)$ *is* **convex in** $x \in \mathcal{X}$ *when* $y \in \mathcal{Y}$ *as well as concave* $y \in \mathcal{Y}$ *when* $x \in \mathcal{X}$.

Comparison: Rockafellar's variational sufficiency (cont'd)

Theorem (Rockafellar, MP 21')

With respect to $\bar{x}$ and $\bar{y}$ satisfying the first-order optimality condition, the **variational sufficiency** *for local optimality holds if and only if, for $\rho > 0$ sufficiently large, there is a closed convex neighborhood $\mathcal{X} \times \mathcal{Y}$ of $(\bar{x}, \bar{y})$ such that $\mathcal{L}(x, y, \rho)$ is* **convex in $x \in \mathcal{X}$** *when $y \in \mathcal{Y}$ as well as concave $y \in \mathcal{Y}$ when $x \in \mathcal{X}$.*

Theorem (Rockafellar, MP 21')

The **strongly variational sufficiency** *for local optimality implies* **augmented tilt stability**. *It corresponds equally to having the functions $\mathcal{L}(\cdot, y, \rho)$ on $\mathcal{X}$ for $y \in \mathcal{Y}$ be* **strongly convex**, *all with the same modulus of strong convexity.*

Both may fail easily even under SOSC!

The proof is in the pudding

Proposition

Let $(\bar{x}, \bar{\lambda})$ be a KKT point satisfying SOSC. Define

$$\mathcal{S}_\rho(\lambda) = \arg\min\{\mathcal{L}(x, \lambda, \rho) \mid x \in \mathbb{B}_{\widehat{r}}(\bar{x})\}.$$

Then there are positive constants $\tau, \rho_3, \widehat{r} > 0$ such that for every $\rho \geq \rho_3$ and $\lambda \in \mathbb{B}_{\widehat{r}/2\tau}(\bar{\lambda})$, the set $\mathcal{S}_\rho(\lambda)$ satisfies the uniform isolated calmness property, i.e.,

$$\mathcal{S}_\rho(\lambda) \subseteq \{\bar{x}\} + \tau\|\lambda - \bar{\lambda}\|\mathbb{B}$$

and satisfies $\varnothing \neq \mathcal{S}_\rho(\lambda) \subseteq \operatorname{int}\mathbb{B}_{\widehat{r}}(\bar{x})$.

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and satisfies $\varnothing \neq \mathcal{S}_\rho(\lambda) \subseteq \operatorname{int} \mathbb{B}_{\widehat{r}}(\bar{x})$.

- (Sun at al., **MP** 08'): nondegeneracy + strong SOSC $\Longrightarrow$ Lipschitz continuity
- (Rockafellar, **MP** 21'):
  strongly variational sufficiency $\Longrightarrow$ augmented tilt stability
  - ⋆ stronger than SOSC
  - ⋆ fully amenable function

  works for NLP, NLSOC; but **NLSDP** is **another kettle of fish**

Proof sketch

A refined perturbation analysis of eigenvalue decompositions

$\Downarrow$

SOSC $+$ **Uniform second Expansion of Moreau envelop**

Proof sketch

A refined perturbation analysis of eigenvalue decompositions

$$\Downarrow$$

SOSC $+$ Uniform second Expansion of Moreau envelop $\Longrightarrow$ Uniform quadratic growth condition

Proof sketch

A refined perturbation analysis of eigenvalue decompositions

$\Downarrow$

SOSC $+$ **Uniform second Expansion of Moreau envelop** $\Longrightarrow$ Uniform quadratic growth condition

$\Downarrow$ $+$

Uniform isolated calmness of AL subproblems

Proof sketch

A refined perturbation analysis of eigenvalue decompositions

$\Downarrow$

SOSC + Uniform second Expansion of Moreau envelop $\Longrightarrow$ Uniform quadratic growth condition

$\Downarrow$ +

Uniform isolated calmness of AL subproblems + Semi-isolated calmness + Assumption

Proof sketch

A refined perturbation analysis of eigenvalue decompositions

⇓

SOSC + Uniform second Expansion of Moreau envelop ⟹ Uniform quadratic growth condition

⇓ (from SOSC)

\+ (below Uniform quadratic growth condition)

Uniform isolated calmness of AL subproblems + Semi-isolated calmness + Assumption

⇓

Local linear convergence of ALM

Semi-isolated calmness

Definition

The semi-isolated clamness for the mapping S_{KKT} at $((0,0,0),(\bar{x},\bar{y},\overline{\Gamma}))$ holds if there exists $\kappa > 0$ and open neighborhoods $\mathbb{V}$ of $(0,0,0)$ and $\mathbb{U}$ of $(\bar{x},\bar{y},\overline{\Gamma})$ such that for all $(x,y,\Gamma) \in S_{\mathrm{KKT}}(a_1,a_2,b) \cap \mathbb{U}$,

$$\|x-\bar{x}\| + \mathrm{dist}((y,\Gamma),\mathcal{M}(\bar{x})) \leq \kappa_1 \|(a_1,a_2,b)\|$$

Sufficient conditions for semi-isolated calmness

Theorem

Let $\bar{x} \in \mathcal{X}$ *be a locally optimal solution to the NLSDP and* $(\bar{y}, \overline{\Gamma}) \in \mathcal{M}(\bar{x})$*. Suppose SOSC holds at* $(\bar{x}, \bar{y}, \overline{\Gamma})$ *and*

$$\mathcal{G}_1(\bar{x}) \cap \mathrm{ri}\, \mathcal{G}_2(\bar{x}) \neq \varnothing,$$

where $\mathcal{G}_1(\bar{x}) = \{(y, \Gamma) \in \mathcal{Y} \times \mathcal{S}^n \mid \nabla f(\bar{x}) + \nabla h(\bar{x})^* y + \nabla G(\bar{x})^* \Gamma = 0\}$ *and* $\mathcal{G}_2(\bar{x}) = \{(y, \Gamma) \in \mathcal{Y} \times \mathcal{S}^n \mid \Gamma \in \mathcal{N}_{\mathcal{S}^n_+}(G(\bar{x}))\}$*. Then there exist a constant* $\kappa_1 > 0$ *and a neighborhood* $\mathbb{U} := \mathbb{B}_{r_1}(\bar{x}, \bar{y}, \overline{\Gamma})$ *of* $(\bar{x}, \bar{y}, \overline{\Gamma})$ *such that for any* $(a_1, a_2, b) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{S}^n$*,*

$$\|x - \bar{x}\| + \mathrm{dist}((y, \Gamma), \mathcal{M}(\bar{x})) \leq \kappa_1 \|(a_1, a_2, b)\| \quad \forall\, (x, y, \Gamma) \in S_{\mathrm{KKT}}(a_1, a_2, b) \cap \mathbb{U}.$$

Sufficient conditions for semi-isolated calmness

Theorem

Let $\bar{x} \in \mathcal{X}$ be a locally optimal solution to the NLSDP and $(\bar{y}, \overline{\Gamma}) \in \mathcal{M}(\bar{x})$. Suppose SOSC holds at $(\bar{x}, \bar{y}, \overline{\Gamma})$ and

$$\mathcal{G}_1(\bar{x}) \cap \operatorname{ri} \mathcal{G}_2(\bar{x}) \neq \varnothing,$$

where $\mathcal{G}_1(\bar{x}) = \{(y, \Gamma) \in \mathcal{Y} \times \mathcal{S}^n \mid \nabla f(\bar{x}) + \nabla h(\bar{x})^* y + \nabla G(\bar{x})^* \Gamma = 0\}$ and $\mathcal{G}_2(\bar{x}) = \{(y, \Gamma) \in \mathcal{Y} \times \mathcal{S}^n \mid \Gamma \in \mathcal{N}_{\mathcal{S}^n_+}(G(\bar{x}))\}$. Then there exist a constant $\kappa_1 > 0$ and a neighborhood $\mathbb{U} := \mathbb{B}_{r_1}(\bar{x}, \bar{y}, \overline{\Gamma})$ of $(\bar{x}, \bar{y}, \overline{\Gamma})$ such that for any $(a_1, a_2, b) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{S}^n$,

$$\|x - \bar{x}\| + \operatorname{dist}((y, \Gamma), \mathcal{M}(\bar{x})) \leq \kappa_1 \|(a_1, a_2, b)\| \quad \forall\, (x, y, \Gamma) \in S_{\mathrm{KKT}}(a_1, a_2, b) \cap \mathbb{U}.$$

Other sufficient conditions:

- SOSC + SRCQ, reduced to isolated calmness
- SSOSC + nondegeneracy, reduced to strong regularity

Assumption: For all $\lambda = (y, \Gamma) \notin \mathcal{M}(\bar{x})$ sufficiently close to $\bar{\lambda} \in \operatorname{rbd} \mathcal{M}(\bar{x})$ and x sufficiently close to $\bar{x}$, there also exists $\widehat{\lambda} \in \mathcal{M}(\bar{x})$ with $\pi(\widehat{\Gamma}) = \pi(\overline{\Gamma})$ such that

$$\|\Pi_{\mathcal{M}(\bar{x})}(\lambda) - \widehat{\lambda}\| = O(R(x, \lambda)).$$

Example 1:

$$\begin{array}{ll} \min & \frac{1}{2}x^3 \\ \text{s.t} & -x^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}^3_+ \quad \Leftarrow \Gamma \end{array}$$

The optimal solution is $\bar{x} = 0$ and $\mathcal{M}(\bar{x}) = \{\Gamma \mid \Gamma \in \mathcal{S}^3_-\}$. The SOSC and bounded linear regular hold at $(\bar{x}, \overline{\Gamma})$, trivially.

Pick $\overline{\Gamma} = \operatorname{Diag}(0, -1, -2)$. Then for all $\Gamma \in \mathbb{B}_{\min\{1/2, r_1\}}(\overline{\Gamma}) \backslash \mathcal{M}(\bar{x})$,

$$\Pi_{\mathcal{M}(\bar{x})}(\Gamma) = Q\operatorname{Diag}(\min\{0, \Gamma_1\}, \Gamma_2, \Gamma_3)Q^T,$$

where $Q \in \mathcal{O}^3(\Gamma)$. Let $\widehat{\Gamma} = Q\operatorname{Diag}(0, \Gamma_2, \Gamma_3)Q^T$. By the bounded linear regular property, we have

$$\|\Pi_{\mathcal{M}(\bar{x})}(\Gamma) - \widehat{\Gamma}\| = \operatorname{dist}(\Gamma, \mathcal{M}(\bar{x})) = O(R(x, \Gamma)).$$

Example 2:

$$\begin{array}{ll} \min & \frac{1}{2}x^2 + 2t \\ \text{s.t} & tA - x^2 I_2 \in \mathcal{S}^2_+, \quad \Leftarrow \Gamma \\ & t \geq 0, \qquad\qquad \Leftarrow y \end{array}$$

where $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$. This problem possesses the unique optimal solution $(\bar{t}, \bar{x}) = (0, 0)$. The corresponding multiplier is

$$\mathcal{M}(\bar{t}, \bar{x}) = \{(\Gamma, y) \in \mathcal{S}^2_- \times \Re \mid \langle A, -\Gamma \rangle \leq 2\}.$$

We can pick $\overline{\Gamma} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. It is easy to see that $\begin{bmatrix} -1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \in \mathcal{G}_1(\bar{t}, \bar{x}) \cap \mathrm{ri}\, \mathcal{G}_2(\bar{t}, \bar{x})$ and the SOSC holds at $(\bar{t}, \bar{x}, \overline{\Gamma})$.

For all $\Gamma \in \mathbb{B}_{\min\{r_1, 1/(2\sqrt{10})\}}(\overline{\Gamma})$, we know that $\langle A, -\Gamma \rangle < 2$, which implies that $\Pi_{\mathcal{M}(\bar{t}, \bar{x})}(\Gamma) = \Pi_{\mathcal{S}^2_-}(\Gamma)$. For $\Gamma = Q\mathrm{Diag}(\Gamma_1, \Gamma_2)Q^T$ with $Q \in \mathcal{O}^2(\Gamma)$, let $\widehat{\Gamma} = Q\mathrm{Diag}(0, \Gamma_2)Q^T$. $\|\Pi_{\mathcal{M}(\bar{x})}(\Gamma) - \widehat{\Gamma}\| = \mathrm{dist}(\Gamma, \mathcal{M}(\bar{x})) = O(R(x, \Gamma))$, which verifies Assumption.

Conclusions

In this talk:

- Convergence rate of ALM for NLSDP without the solution uniqueness assumption
- Uniformly second order expansion for Moreau envelop of SDP
- Sufficient conditions for semi-isolated calmness

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- Convergence rate of ALM for NLSDP without the solution uniqueness assumption
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Future work:

- Implementable stopping criterion
- dual Q-linear + primal R-linear

谢谢大家!